Weyl Transforms for H-Type Groups*

Aparajita Dasgupta
École Polytechnique Fédérale de Lausanne, Mathematics
Building, Section 8, CH-1015, Lausanne, Switzerland
e-mail: aparajita.dasgupta@epfl.ch

M. W. Wong
Department of Mathematics and Statistics, York University,
4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
e-mail: mwwong@mathstat.yorku.ca

Abstract Weyl transforms for H-type groups are introduced and shown to be the classical Weyl transforms on $\mathbb{R}^n$.

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1 Introduction

In [8] among others, pseudo-differential operators on $\mathbb{R}^n$ are built on the Fourier inversion formula for the Fourier transform on $\mathbb{R}^n$. The Fourier transform on the Heisenberg group is defined in terms of the Schrödinger

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representations of the Heisenberg group on $L^2(\mathbb{R}^n)$ and the Fourier inversion formula for the Fourier transform on the Heisenberg group can then be used to define pseudo-differential operators with operator-valued symbols on the Heisenberg group. A key technique in studying pseudo-differential operators on the Heisenberg group in [2] is to express the Fourier transform on the Heisenberg group in terms of Weyl transforms on $\mathbb{R}^n$, which has been studied extensively in [6]. The aim of this paper is to look at the Fourier transform on H-type groups based on the Schrödinger representations of H-type groups on $L^2(\mathbb{R}^n)$ explained in the appendix of [4] and prove that Fourier transforms on H-type groups are Weyl transforms on $\mathbb{R}^n$. As such, the results in [2] can then be formulated *verbatim* on H-type groups.

For the sake of completeness and transparency, we first recall in Section 2 how to set up pseudo-differential operators on the Heisenberg group. H-type groups and their Schrödinger representations on $L^2(\mathbb{R}^n)$ are recapitulated without proofs in Section 3. λ-Weyl transforms for H-type groups are introduced in Section 4 and proved to be $|\lambda|$-Weyl transforms for the Heisenberg group. That the Fourier transform on a H-type group is a classical Weyl transform on $\mathbb{R}^n$ is proved in Section 5.

2 The Heisenberg Group

If we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$ via the obvious identification

$$\mathbb{R}^n \times \mathbb{R}^n \ni (q,p) \leftrightarrow q + ip \in \mathbb{C}^n,$$

and we let

$$\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R},$$

then $\mathbb{H}^n$ becomes a noncommutative group when equipped with the multiplication $\cdot$ given by

$$(z,t)\cdot(w,s) = \left(z + w, t + s + \frac{1}{2}\omega(z,w)\right), \quad (z,t),(w,s) \in \mathbb{H}^n,$$

where $\omega(z,w)$ is the symplectic form of

$$z = (z_1, \ldots, z_n)$$

and

$$w = (w_1, \ldots, w_n)$$

defined by

$$\omega(z, w) = \operatorname{Im}(z \cdot \overline{w}) = \operatorname{Im} \sum_{j=1}^{n} z_j \overline{w_j}.$$

In fact, $\mathbb{H}^n$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz\,dt$.

One of the most fundamental problems in the analysis on a Lie group is the classification of all irreducible and unitary representations of the Lie group. To that end for the Heisenberg group $\mathbb{H}^n$, we let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and let $U(L^2(\mathbb{R}^n))$ be the group of all unitary operators on $L^2(\mathbb{R}^n)$. For all $\lambda \in \mathbb{R}^*$, we let $\rho_\lambda : \mathbb{H}^n \to U(L^2(\mathbb{R}^n))$ be the mapping defined by

$$(\rho_\lambda(z, t)f)(x) = e^{i\lambda t} e^{\lambda(iq\cdot x + (iq\cdot p/2))} f(x + p), \quad x \in \mathbb{R}^n,$$

for all f in $L^2(\mathbb{R}^n)$. Then it can be proved that $\rho_\lambda : \mathbb{H}^n \to U(L^2(\mathbb{R}^n))$ is an irreducible and unitary representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$. In fact, the Stone–von Neumann theorem says that these are essentially all the irreducible and unitary representations of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$. More precisely, we have the following Stone–von Neumann theorem.

Theorem 2.1 *If $\rho : \mathbb{H}^n \to U(X)$ is an irreducible and unitary representation of $\mathbb{H}^1$ on X, where $U(X)$ is the group of all unitary operators on an infinite-dimensional, separable and complex Hilbert space X, such that there exists a real number λ in $\mathbb{R}^*$ for which*

$$\rho(0, t) = e^{i\lambda t} I, \quad t \in \mathbb{R},$$

where I is the identity operator on X, then ρ is unitarily equivalent to ρ_λ in the sense that there exists a bijective isometry $U : X \to L^2(\mathbb{R}^n)$ such that

$$\rho(z, t) = U^{-1} \rho_\lambda(z, t) U, \quad (z, t) \in \mathbb{H}^n.$$

Henceforth, the identification of $\{\rho_\lambda : \lambda \in \mathbb{R}^*\}$ with $\mathbb{R}^*$ will be used. Now, let $f \in L^1(\mathbb{H}^n)$ and let $\lambda \in \mathbb{R}^*$. Then we define the Fourier transform $\hat{f}(\lambda)$ of f at λ to be the bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ given by

$$\hat{f}(\lambda)\varphi = \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} f(z, t) \rho_\lambda(z, t) \varphi \, dz \, dt$$

for all $\varphi \in L^2(\mathbb{R}^n)$. In fact, the following result is valid.

Theorem 2.2 *Let* $f \in L^2(\mathbb{H}^n)$ *and let* $\lambda \in \mathbb{R}^*$. *Then* $\hat{f}(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a Hilbert–Schmidt operator. In fact,*

$$\int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{S_2}^2 d\mu(\lambda) = \|f\|_{L^2(\mathbb{H}^n)}^2,$$

where $\| \|_{S_2}$ *stands for the norm in the Hilbert space* S_2 *of all Hilbert–Schmidt operators on* $L^2(\mathbb{R}^n)$ *and* $d\mu(\lambda) = (2\pi)^{-(n+1)}|\lambda|^n d\lambda$.

Remark 2.3 The formula in Theorem 2.2 is known as Plancherel's formula and the measure $d\mu(\lambda)$ is called the Plancherel measure on the Heisenberg group.

The starting point for the analysis of pseudo-differential operators on the Heisenberg group in [2] is the following Fourier inversion formula for the Fourier transform on the Heisenberg group.

Theorem 2.4 *Let* f *be a function in the Schwartz space* $\mathcal{S}(\mathbb{H}^n)$ *on* $\mathbb{H}^n$. *Then for all* $(z,t) \in \mathbb{H}^n$,

$$f(z,t) = \int_{-\infty}^{\infty} \operatorname{tr}(\rho_\lambda(z,t)^* \hat{f}(\lambda)) d\mu(\lambda),$$

where $\rho_\lambda(z,t)^*$ *is the adjoint of* $\rho_\lambda(z,t)$.

The theory of the Heisenberg group hitherto described can be found in many places, e.g., in [5, 7].

Let $B(L^2(\mathbb{R}^n))$ be the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$. Then we call a mapping $\sigma : \mathbb{H}^n \times \mathbb{R}^* \to B(L^2(\mathbb{R}^n))$ an operator-valued symbol or simply a symbol. Given a symbol $\sigma : \mathbb{H}^n \times \mathbb{R}^* \to B(L^2(\mathbb{R}^n))$, we define the pseudo-differential operator $T_\sigma : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ by

$$(T_\sigma f)(z,t) = \int_{-\infty}^{\infty} \operatorname{tr}(\rho_\lambda^*(z,t)\sigma(z,t,\lambda)\hat{f}(\lambda))\, d\mu(\lambda), \quad (z,t) \in \mathbb{H}^n,$$

for all f in $\mathcal{S}(\mathbb{H}^n)$.

Results closely related to the paper [2] and the results in this paper are in [3].

3 H-Type Groups

Let $\mathfrak{g}$ be a $(2n+m)$-dimensional real Lie algebra equipped with a Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Suppose that there exists an inner product $(\ ,\ )$ in $\mathfrak{g}$ such that

$$[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z},$$

where $\mathfrak{z}^\perp$ is the orthogonal complement of $\mathfrak{z}$, and for every nonzero λ in $\mathfrak{z}$, the mapping $J_\lambda : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$(J_\lambda V, W) = (\lambda, [V, W]), \quad V, W \in \mathfrak{z}^\perp,$$

is orthogonal whenever $(\lambda, \lambda) = 1$. Then we call $\mathfrak{g}$ a H-type Lie algebra. It is easy to check that for every nonzero element λ in $\mathfrak{z}$,

$$J_\lambda^2 = -I.$$

A H-type group $\mathbb{G}$ is a connected and simply connected Lie group $\mathbb{G}$ such that the corresponding Lie algebra $\mathfrak{g}$ is a H-type Lie algebra.

It can be proved [1] that $\mathbb{G}$ is a H-type group if and only if $\mathbb{G}$ is isomorphic to $\mathbb{R}^{2n} \times \mathbb{R}^m$ equipped with the group law $\cdot$ given by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{2}\omega(z, w) \right), \quad (z, t), (w, s) \in \mathbb{G},$$

where $\omega(z, w)$ is a point in $\mathbb{R}^m$ of which the j^{th} entry is $(U^{(j)} z, w)$ and $U^{(1)}, \ldots, U^{(m)}$ are $2n \times 2n$ skew-symmetric and orthogonal matrices such that

$$U^{(j)} U^{(k)} + U^{(k)} U^{(j)} = 0$$

for all $j, k = 1, \ldots, m$ with $j \neq k$.

Let $\mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}$. Then for $\lambda \in \mathbb{R}^{m*}$, let $R_\lambda \in \mathrm{O}(2n, \mathbb{R})$ be such that

$$J_\lambda = R_\lambda J R_\lambda^t,$$

where J is the symplectic matrix of order $2n \times 2n$ given by

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Here I_n denotes the identity matrix of order n and R_λ^t is the transpose of R_λ. The following lemma in [4] establishes a useful link between the representations of $\mathbb{G}$ and those of the Heisenberg group $\mathbb{H}^n$.

Lemma 3.1 *The mapping $\alpha_\lambda : \mathbb{G} \to \mathbb{H}^n$ given by*

$$\alpha_\lambda(z,t) = \left(R_\lambda^t z, \frac{\lambda \cdot t}{|\lambda|} \right), \quad (z,t) \in \mathbb{R}^{2n} \times \mathbb{R}^m,$$

is a surjective homomorphism of Lie groups. In particular, $\mathbb{G}/\ker \alpha_\lambda$ is isomorphic to $\mathbb{H}^n$.

In fact,

$$\ker \alpha_\lambda = \left\{ (z,t) \in \mathbb{G} : \left(R_\lambda^t z, \frac{\lambda \cdot t}{|\lambda|} \right) = (0,0) \right\} = \{(0,t) \in \mathbb{G} : t \perp \lambda\}.$$

Let $\lambda \in \mathbb{R}^{m*}$. Then we define the irreducible and unitary representation π_λ of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ by

$$\pi_\lambda = \rho_{|\lambda|} \circ \alpha_\lambda.$$

It is then obvious that $\pi_\lambda(0,t) = e^{i\lambda \cdot t} I$. In fact, any irreducible and unitary representation of $\mathbb{G}$ with central character $e^{i\lambda \cdot t}$ factors through the kernel of α_λ and hence by the Stone-von Neumann theorem must be equivalent to π_λ. The representation π_λ of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ is called the Schrödinger representation. So, for $\lambda \in \mathbb{R}^{m*}$, the Schrödinger representation π_λ of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ is given by

$$(\pi_\lambda(z,t)h)(x) = \left(\rho_{|\lambda|} \left(R_\lambda^t z, \frac{\lambda \cdot t}{|\lambda|} \right) h \right)(x), \quad x \in \mathbb{R}^n,$$

for all h in $L^2(\mathbb{R}^n)$.

For $\lambda \in \mathbb{R}^{m*}$, let

$$R_\lambda = [R_{\lambda,1}, \ldots, R_{\lambda,2n}],$$

where $R_{\lambda,j}$ is a $2n \times 1$ matrix for $j = 1, \ldots, 2n$. Then

$$R_\lambda^t = \begin{bmatrix} R_{\lambda,1}^t \\ \vdots \\ R_{\lambda,2n}^t \end{bmatrix}.$$

Let $R_{\lambda,j}^t z = q_{\lambda,j}$, $j = 1, \ldots, n$, and $R_{\lambda,j}^t z = p_{\lambda,j-n}$, $j = n+1, \ldots, 2n$. Then for all h in $L^2(\mathbb{R}^n)$

$$(\pi_\lambda(z,t)h)(x) = e^{i|\lambda|\left(\frac{\lambda \cdot t}{|\lambda|} + q_\lambda \cdot x + \frac{1}{2} q_\lambda \cdot p_\lambda \right)} h(x + p_\lambda), \quad x \in \mathbb{R}^n,$$

where $q_\lambda = (q_{\lambda,1}, \ldots, q_{\lambda,n})$ and $p_\lambda = (p_{\lambda,1}, \ldots, p_{\lambda,n})$.

The set $\{\pi_\lambda : \lambda \in \mathbb{R}^{m*}\}$ of irreducible and unitary representations of the H-type group $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ can best be identified with the *punctured* Euclidean space $\mathbb{R}^{m*}$.

Let $f \in L^1(\mathbb{G})$ and let $\lambda \in \mathbb{R}^{m*}$. Then we define the Fourier transform $\hat{f}(\lambda)$ of f at λ to be the bounded linear operator on $L^2(\mathbb{R}^n)$ by

$$\hat{f}(\lambda)\varphi = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} f(z,t)\pi_\lambda(z,t)\varphi \, dz \, dt, \quad \varphi \in L^2(\mathbb{R}^n).$$

We have the fact that if $f \in L^2(\mathbb{G})$, then for all $\lambda \in \mathbb{R}^{m*}$, $\hat{f}(\lambda)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^n)$ and we have the Plancherel formula to the effect that

$$\|f\|^2_{L^2(\mathbb{G})} = \int_{\mathbb{R}^m} \|\hat{f}(\lambda)\|^2_{S^2} \, d\mu(\lambda), \quad f \in L^2(\mathbb{G}),$$

where $\| \, \|_{S^2}$ denotes the Hilbert–Schmidt norm and $d\mu$ is the Plancherel measure on $\mathbb{G}$ given by

$$d\mu(\lambda) = c|\lambda|^n d\lambda.$$

and c is a suitable normalizing constant. The Fourier inversion formula is given by

$$f(z,t) = \int_{\mathbb{R}^m} \mathrm{tr}(\pi_\mu(z,t)^* \hat{f}(\lambda)) \, d\mu(\lambda), \quad (z,t) \in \mathbb{G},$$

for all Schwartz functions on $\mathbb{G}$, where $\pi_\mu(z,t)^*$ is the adjoint of $\pi_\mu(z,t)$.

Now, let $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to B(L^2(\mathbb{R}^n))$ be an operator-valued symbol or simply a symbol. Then we define the pseudo-differential operator T_σ with symbol σ by

$$(T_\sigma f)(z,t) = \int_{\mathbb{R}^m} \mathrm{tr}(\pi_\mu(z,u)^* \sigma(z,t,\lambda)\hat{f}(\lambda)) \, d\mu(\lambda), \quad (z,t) \in \mathbb{G},$$

for all $f \in \mathcal{S}(\mathbb{G})$.

4 Weyl Transforms for H-Type Groups

Let q and p be in $\mathbb{R}^n$, and let $\lambda \in \mathbb{R}^{m*}$. Then for every measurable function f on $\mathbb{R}^n$, the function $\pi_\lambda(q,p)f$ on $\mathbb{R}^n$ is defined by

$$(\pi_\lambda(q,p)f)(x) = e^{i|\lambda|(q_\lambda \cdot x + (q_\lambda \cdot p_\lambda/2))} f(x + p_\lambda), \quad x \in \mathbb{R}^n,$$

where $q_\lambda = (q_{\lambda,1}, \ldots, q_{\lambda,n})$ and $p_\lambda = (p_{\lambda,1}, \ldots, p_{\lambda,n})$. It is clear that $\pi_\lambda(q,p) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator for all q and p in $\mathbb{R}^n$.

Let f and g be in $L^2(\mathbb{R}^n)$. Then we define the function $V_\lambda(f,g)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$V_\lambda(f,g)(q,p) = (2\pi)^{-n/2}(\pi_\lambda(q,p)f, g)_{L^2(\mathbb{R}^n)}.$$

We call $V_\lambda(f,g)$ the λ-Fourier–Wigner transform of f and g and the λ-Wigner transform $W_\lambda(f,g)$ of f and g is defined by

$$W_\lambda(f,g) = V_\lambda(f,g)^\wedge,$$

where $V_\lambda(f,g)^\wedge$ is the Fourier transform of $V_\lambda(f,g)$.

Let σ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then for $\lambda \in \mathbb{R}^{m*}$, we define the λ-Weyl transform with symbol σ to be the bounded linear operator $W_\sigma^\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$(W_\sigma^\lambda f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi) W_\lambda(f,g)(x,\xi)\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$. Using the adjoint formula in Fourier analysis, we get

$$(W_\sigma^\lambda f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) V_\lambda(f,g)(q,p)\, dq\, dp$$

for all f and g in $L^2(\mathbb{R}^n)$. We can also write *formally*

$$\begin{aligned} W_\sigma^\lambda &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) \pi_\lambda(q,p)\, dq\, dp \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) \pi_\lambda(z)\, dq\, dp \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(z) \rho_{|\lambda|}(R_\lambda^t z)\, dq\, dp \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(R_\lambda z) \rho_{|\lambda|}(z)\, dq\, dp. \end{aligned}$$

Since

$$\hat{\sigma}(R_\lambda z) = (\sigma \circ R_\lambda)^\wedge(z), \quad z \in \mathbb{C}^n,$$

it follows that

$$W_\sigma^\lambda = W_{\sigma \circ R_\lambda}^{|\lambda|}.$$

So, for any unit vector u in $\mathbb{R}^{m*}$,

$$W_\sigma^u = W_{\sigma\circ R_u},$$

which is the classical Weyl transform in [6].

5 The Main Result

We prove in this section that the Fourier transform on a H-type group is in fact a classical Weyl transform on $\mathbb{R}^n$.

Theorem 5.1 *Let $f \in L^1(\mathbb{G})$. Then for all $\lambda \in \mathbb{R}^{m*}$,*

$$\hat{f}(\lambda) = (2\pi)^{(2n+m)/2} W_{D^1_{|\lambda|}((f^\lambda)^\vee)\circ R_\lambda},$$

where $D^1_{|\lambda|}$ is the dilation operator defined by

$$(D^1_{|\lambda|}w)(q,p) = w(|\lambda|q,p), \quad q,p \in \mathbb{R}^n,$$

for all measurable functions w on $\mathbb{R}^n\times\mathbb{R}^n$, and f^λ is the function on $\mathbb{R}^n\times\mathbb{R}^n$ defined by

$$f^\lambda(z) = (2\pi)^{-m/2}\int_{\mathbb{R}^m} e^{i\lambda\cdot t} f(z,t)\,dt, \quad z \in \mathbb{C}^n,$$

and $h^\vee$ denotes the inverse Fourier transform of the function h on $\mathbb{C}^n$.

Proof Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then for almost all λ in $\mathbb{R}^{m*}$,

$$\begin{aligned}
&(\hat{f}(\lambda)\varphi)(x)\\
= {}& \int_{\mathbb{R}^m}\int_{\mathbb{C}^n} f(z,t)(\pi_\lambda(z,t)\varphi)(x)\,dz\,du\\
= {}& \int_{\mathbb{R}^m}\int_{\mathbb{C}^n} f(z,t)e^{it\cdot\lambda}(\rho_{|\lambda|}(R^t_\lambda z)\varphi)(x)\,dz\,dt\\
= {}& (2\pi)^{m/2}\int_{\mathbb{C}^n} f^\lambda(R_\lambda z)(\rho_{|\lambda|}(z)\varphi)(x)\,dz\\
= {}& (2\pi)^{m/2}\int_{\mathbb{C}^n} ((f^\lambda)^\vee)^\wedge(R_\lambda(q,p))e^{i|\lambda|(q\cdot x+(q\cdot p/2))}\varphi(x+p)\,dq\,dp.
\end{aligned}$$

Then

$$(\hat{f}(\lambda)\varphi)(x) = (2\pi)^{m/2}\int_{\mathbb{C}^n}(D^1_{|\lambda|}((f^\lambda)^\vee)^\wedge(R_\lambda(q,p))(\rho(q,p)\varphi)(x)\,dq\,dp.$$

Thus, for all $\lambda \in \mathbb{R}^{m*}$,

$$\hat{f}(\lambda) = (2\pi)^{(2n+m)/2} W_{D^1_{|\lambda|}((f^\lambda)^\vee)\circ R_\lambda}.$$

□

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