Characterizations of Nuclear Pseudo-Differential Operators on $\mathbb{S}^1$ with Applications to Adjoints and Products

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Abstract We give necessary and sufficient conditions on the symbols to guarantee that the corresponding pseudo-differential operators are nuclear from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ for $1 \le p_1, p_2 < \infty$. Applications are given to adjoints of nuclear pseudo-differential operators from $L^{p_2'}(\mathbb{S}^1)$ into $L^{p_1'}(\mathbb{S}^1)$ for $1 \le p_1, p_2 < \infty$ and products of nuclear pseudo-differential operators on $L^p(\mathbb{S}^1)$, $1 \le p < \infty$.

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1 Introduction

Nuclear operators on Banach spaces as generalizations of trace class operators can be traced at least to Grothendieck [10, 11]. First results on nuclear integral operators and pseudo-differential operators on L^p spaces in simple settings like the unit circle centered at the origin and the discrete group of all integers, $1 \leq p < \infty$, can be found in [2, 3, 6].

Let $\mathbb{S}^1$ be the unit circle centered at the origin and let $\mathbb{Z}$ be the set of all integers. For every measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ and every measurable function f on $\mathbb{S}^1$, we define the function $T_\sigma f$ on $\mathbb{S}^1$ *formally* by

$$(T_\sigma f)(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \sigma(\theta, n) \hat{f}(n), \quad \theta \in [-\pi, \pi],$$

where $\hat{f}(n)$ is the Fourier transform of f given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

We call T_σ the pseudo-differential operator on $\mathbb{S}^1$ with symbol σ.

Conditions on the symbols σ to insure the boundedness, compactness and self-adjointness of the corresponding pseudo-differential operators T_σ have been given in [1, 7, 8, 12, 13, 14, 15, 17]. In addition, Fredholmness and nuclearity of pseudo-differential operators T_σ under suitable conditions on the symbols σ are investigated in [2, 3]. These results can be extended easily from the unit circle $\mathbb{S}^1$ to the n-torus $\mathbb{T}^n$ given by

$$\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}.$$

Extensions to compact Lie groups and to compact manifolds can be found, for instance, in, respectively, [5] and [4] In a nutshell, the results hitherto cited are on sufficient conditions on the symbols σ to prove mapping properties of the corresponding pseudo-differential operators T_σ. To the best of our knowledge, very few results on necessary and sufficient conditions on the symbols σ for the corresponding pseudo-differential operators to have the desired mapping properties exist. A notable exception is the paper [12] by Molahajloo on compact pseudo-differential operators on $\mathbb{S}^1$.

The aim of this paper is to present necessary and sufficient conditions on the symbols σ for the corresponding pseudo-differential operators T_σ to be nuclear from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ for $1 \le p_1, p_2 < \infty$. We also give necessary and sufficient conditions on the symbols σ to guarantee that the adjoints and products of pseudo-differential operators are nuclear.

We first give in Section 2 the definition of nuclear operators on Banach spaces and the main tool [2, 3, 6] to be used in this paper. Then we give necessary and sufficient conditions on the symbols σ so that the corresponding pseudo-differential operators T_σ are nuclear from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ for $1 \le p_1, p_2 < \infty$. In Section 3 are given necessary and sufficient conditions on the symbols σ for which the adjoints of pseudo-differential operators T_σ are nuclear. The nuclearity of products of nuclear operators is given in Section 4.

All results in this paper can be routinely extended from the unit circle $\mathbb{S}^1$ to the n-torus $\mathbb{T}^n$. It is worth pointing out that characterizing trace class pseudo-differential operators on $L^2(\mathbb{R}^n)$ can be found in [16].

2 Nuclearity on $L^p(\mathbb{S}^1)$

Let X and Y be complex Banach spaces and let $T : X \to Y$ be a bouned linear operator. Suppose that we can find sequences $\{x'_n\}_{n=1}^\infty$ in the dual space X' of X and $\{y_n\}_{n=1}^\infty$ in Y such that

$$\sum_{n=1}^\infty \|x'_n\|_{X'} \|y_n\|_Y < \infty$$

and

$$Tx = \sum_{n=1}^\infty x'_n(x) y_n, \quad x \in X.$$

Then we call $T : X \to Y$ a *nuclear operator* and its *trace* $\mathrm{tr}(T)$ is defined by

$$\mathrm{tr}(T) = \sum_{n=1}^\infty x'_n(y_n).$$

It can be proved that the definition of a nuclear operator and the definition of the trace of a nuclear operator are independent of the choices of the sequences $\{x'_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$. Details can be found in [9].

For L^p spaces, the main tool is the following result in [2, 3, 6].

Theorem 2.1 *Let (X_1, μ_1) and (X_2, μ_2) be σ-finite measure spaces. A bounded linear operator $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$, $1 \le p_1, p_2 < \infty$, is nuclear if and only if there exist sequences $\{g_n\}_{n=1}^{\infty}$ in $L^{p_1'}((X_1, \mu_1)$ and $\{h_n\}_{n=1}^{\infty}$ in $L^{p_2}(X_2, \mu_2)$ such that for all $f \in L^{p_1}(X_1, \mu_1)$,*

$$(Tf)(x) = \int_{X_1} \left(\sum_{n=1}^{\infty} h_n(x) g_n(y) \right) f(y)\, d\mu_1(y), \quad x \in X_2,$$

and

$$\sum_{n=1}^{\infty} \|g_n\|_{L^{p_1'}(X_1,\mu_1)} \|h_n\|_{L^{p_2}(X_2,\mu_2)} < \infty.$$

The functin K on $X_2 \times X_1$ defined by

$$K(x, y) = \sum_{n=1}^{\infty} h_n(x) g_n(y), \quad x \in X_2, y \in X_1,$$

is known as the kernel of the nuclear operator $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$. If $X_1 = X_2 = X$, $p_1 = p_2 = p$ and $\mu_1 = \mu_2 = \mu$ is a σ-finite measure, then for almost all $x \in X$,

$$\int_X |K(x, y)|\, d\mu(y) \le \sum_{n=1}^{\infty} \|h_n\|_{L^p(X,\mu)} \|g_n\|_{L^{p'}(X,\mu)}.$$

The trace $\mathrm{tr}(T)$ of $T : L^p(X, \mu) \to L^p(X, \mu)$ is given by

$$\mathrm{tr}(T) = \int_X K(x, x)\, d\mu(x).$$

We give in the following theorem a necessary and sufficent condition on the symbol σ to make sure that the corresponding pseudo-differential operator T_σ from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ is nuclear for $1 \le p_1, p_2 < \infty$.

Theorem 2.2 *Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then the pseudo-differential operator $T_\sigma : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is nuclear for $1 \le p_1, p_2 < \infty$ if* *and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that*

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{S}^1)} \|g_k\|_{L^{p'_1}(\mathbb{S}^1)} < \infty$$

and

$$\sigma(\theta, n) = 2\pi e^{-in\theta} \sum_{k=-\infty}^{\infty} h_n(\theta) \widehat{g_k}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

Proof We only need to prove the necessity. Suppose that $T_\sigma : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is nuclear. By Theorem 2.1, there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that

$$\sum_{k=1}^{\infty} \|k_k\|_{L^{p_2}(\mathbb{S}^1)} \|g_k\|_{L^{p'_1}(\mathbb{S}^1)} < \infty$$

and for all $f \in L^{p_1}(\mathbb{S}^1)$,

$$\begin{aligned} (T_\sigma f)(\theta) &= \sum_{k=-\infty}^{\infty} e^{ik\theta} \sigma(\theta, k) \hat{f}(k) \\ &= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} h_k(\theta) g_k(\phi) \right) f(\phi)\, d\phi, \quad \theta \in [-\pi, \pi]. \end{aligned} \tag{2.1}$$

Now, for all $n \in \mathbb{Z}$, we let e_n be the function on $\mathbb{S}^1$ defined by

$$e_n(\theta) = e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Since

$$\widehat{e_n}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} e^{in\theta} d\theta = \begin{cases} 0, & k \neq n, \\ 1, & k = n. \end{cases}$$

If we let $f = e_n$ in (2.1), then we get

$$\begin{aligned} e^{in\theta}\sigma(\theta,n) &= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\phi)\right)e^{in\phi}d\phi \\ &= \sum_{k=-\infty}^{\infty} h_k(\theta)\int_{-\pi}^{\pi} e^{in\phi}g_k(\phi)\,d\phi \\ &= 2\pi\sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad \theta\in[-\pi,\pi]. \end{aligned}$$

Therefore

$$\sigma(\theta,n) = 2\pi e^{-in\theta}\sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad (\theta,n)\in\mathbb{S}^1\times\mathbb{Z}.$$

Conversely, suppose that there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that

$$\sum_{k=-\infty}^{\infty}\|h_k\|_{L^{p_2}(\mathbb{S}^1)}\|g_k\|_{L^{p_1'}(\mathbb{S}^1)} < \infty$$

and

$$\sigma(\theta,n) = 2\pi e^{-in\theta}\sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad (\theta,n)\in\mathbb{S}^1\times\mathbb{Z}.$$

Then for all $f \in L^{p_1}(\mathbb{S}^1)$,

$$
\begin{aligned}
(T_\sigma f)(\theta) &= \sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(\theta,n)\hat{f}(n) \\
&= 2\pi \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n)\hat{f}(n) \\
&= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h_k(\theta)\int_{-\pi}^{\pi} e^{in\phi} g_k(\phi)\,d\phi\right)\hat{f}(n) \\
&= \int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} e^{in\phi}\hat{f}(n)\sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\phi)\,d\phi \\
&= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\phi)\right) f(\phi)\,d\phi, \quad \theta\in[-\pi,\pi].
\end{aligned}
$$

□

Before we give an application of Theorem 2.2, we need another characterization of nuclear operators from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$, $1 \le p_1, p_2 < \infty$..

Theorem 2.3 *Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then the pseudo-differential operator $T_\sigma : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is nuclear if and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that*

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{S}^1)}\|g_k\|_{L^{p'}(\mathbb{S}^1)} < \infty$$

and

$$\sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}\sigma(\theta,n) = 4\pi^2 \sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\phi), \quad \theta,\phi\in[-\pi,\pi].$$

Proof Suppose that $T_\sigma : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is nuclear. Then by Theorem 2.1, there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{S}^1)}\|g_k\|_{L^{p_1'}(\mathbb{S}^1)} < \infty$$

and

$$e^{in\theta}\sigma(\theta,n) = 2\pi \sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad (\theta,n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

Thus, for all θ and ϕ in $[-\pi,\pi]$,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}\sigma(\theta,n) &= 2\pi \sum_{n=-\infty}^{\infty} e^{-in\phi} \sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n) \\
&= 2\pi \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} e^{in(\omega-\phi)} \sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\omega)\,d\omega \\
&= 4\pi^2 \int_{-\pi}^{\pi} \delta(\phi-\omega)\left(\sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\omega)\right) d\omega \\
&= 4\pi^2 \sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\phi).
\end{aligned}$$

The converse is clear from Theorem 2.1. □

An immediate consequence of Lemma 2.3 is the following result.

Theorem 2.4 *Let $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ be a nuclear operator, where $1 \le p < \infty$. Then*

$$\operatorname{tr}(T_\sigma) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sigma(\theta,n)\,d\theta.$$

Proof By 2.1 andTheorem 2.3,

$$\operatorname{tr}(T_\sigma) = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h_k(\theta)g_k(\theta)\,d\theta = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sigma(\theta,n)\,d\theta.$$

□

3 Adjoints

Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that the pseudo-differential operator $T_\sigma : L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is nuclear. Then there exist sequences

$\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p_1'}(\mathbb{S}^1)$ and $\{h_k\}_{-\infty}^{\infty}$ in $L^{p_2}(\mathbb{S}^1)$ such that

$$\sum_{k=-\infty}^{\infty} \|g_k\|_{L^{p_1'}(\mathbb{S}^1)} \|h_k\|_{L^{p_2}(\mathbb{S}^1)} < \infty$$

and

$$(T_\sigma f)(\theta) = 2\pi e^{-in\theta} \sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad \theta \in [-\pi, \pi].$$

The following theorem tells us that the adjoint $T_\sigma : L^{p_2'}(\mathbb{S}^1) \to L^{p_1'}(\mathbb{S}^1)$ is nuclear and its symbol σ^* can also be expressed explicitly.

Theorem 3.1 *Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that T_σ : $L^{p_1}(\mathbb{S}^1) \to L^{p_2}(\mathbb{S}^1)$ is nuclear. Let $\{g_k\}_{k=-\infty}^{\infty}$ and $\{h_k\}_{k=-\infty}^{\infty}$ be sequences in, respectively, $L^{p_1'}(\mathbb{S}^1)$ and $L^{p_2}(\mathbb{S}^1)$ such that*

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^{p_2}(\mathbb{S}^1)} \|g_k\|_{L^{p_1'}(\mathbb{S}^1)} < \infty$$

and

$$\sigma(\theta, n) = 2\pi e^{-in\theta} \sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

Then $T_\sigma^* : L^{p_2'}(\mathbb{S}^1) \to L^{p_1'}(\mathbb{S}^1)$ is nuclear and the symbol σ^* of T_σ^* is given by

$$\sigma^*(\theta, n) = 2\pi e^{in\theta} \sum_{m=-\infty}^{\infty} \overline{g_m}(\theta)\widehat{\overline{h_m}}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

Proof For all functions $u \in L^p(\mathbb{S}^1)$ and $v \in L^{p'}(\mathbb{S}^1)$, $1 \le p \le \infty$, we define (u, v) by

$$(u, v) = \int_{-\pi}^{\pi} u(\theta)\overline{v(\theta)}\, d\theta.$$

Now, for all $f \in L^{p_1}(\mathbb{S}^1)$ and $g \in L^{p_2'}(\mathbb{S}^1)$,

$$(T_\sigma f, g) = (f, T_{\sigma^*} g).$$

So,

$$\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} e^{im\theta}\sigma(\theta,m)\hat{f}(m)\overline{g(\theta)}\,d\theta = \int_{-\pi}^{\pi} f(\theta) \sum_{m=-\infty}^{\infty} \overline{e^{im\theta}\sigma^*(\theta,m)\hat{g}(m)}\,d\theta.$$

Now, let $f(\theta) = e^{in\theta}$ and $g(\theta) = e^{ik\theta}$ for all $\theta \in [-\pi,\pi]$, where n and k are integers. Then

$$\int_{-\pi}^{\pi} e^{-i(k-n)\theta}\sigma(\theta,n)\,d\theta = \int_{-\pi}^{\pi} e^{-i(k-n)\theta}\overline{\sigma^*(\theta,k)}\,d\theta.$$

Thus,

$$\overline{\hat{\sigma}(k-n,n)} = \widehat{\sigma^*}(n-k,k), \quad n,k \in \mathbb{Z}.$$

Therefore

$$\begin{aligned}
\sigma^*(\theta,n) &= \sum_{k=-\infty}^{\infty} e^{i(k-n)\theta}\widehat{\sigma^*}(k-n,n) \\
&= \sum_{k=-\infty}^{\infty} e^{i(k-n)\theta}\overline{\hat{\sigma}(n-k,k)} \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{i(k-n)\theta}\overline{\int_{-\pi}^{\pi} e^{-i(n-k)\phi}\sigma(\phi,k)\,d\phi} \\
&= \sum_{k=-\infty}^{\infty} e^{i(k-n)\theta}\overline{\int_{-\pi}^{\pi} e^{-ik\phi}\sum_{m=-\infty}^{\infty} h_m(\phi)\widehat{g_m}(-k)\,d\phi} \\
&= \frac{1}{2\pi}e^{-in\theta}\overline{\int_{-\pi}^{\pi} e^{-in\phi}\sum_{m=-\infty}^{\infty} h_m(\phi)\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} e^{ik(\omega-\theta)}g_m(\omega)\,d\omega\,d\phi} \\
&= e^{-in\theta}\overline{\int_{-\pi}^{\pi} e^{-in\phi}\sum_{m=-\infty}^{\infty} h_m(\phi)\int_{-\pi}^{\pi}\delta(\theta-\omega)g_m(\omega)\,d\omega\,d\phi} \\
&= e^{-in\theta}\overline{\int_{-\pi}^{\pi} e^{-in\phi}\sum_{m=\infty}^{\infty} h_m(\phi)g_m(\theta)\,d\phi} \\
&= 2\pi e^{-in\theta}\sum_{m=-\infty}^{\infty}\overline{g_m}(\theta)\widehat{\overline{h_m}}(-n)
\end{aligned}$$

for all $(\theta,n) \in \mathbb{S}^1 \times \mathbb{Z}$. This completes the proof. □

4 Products

The following theorem tells us in particular that the product of two nuclear operators from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$, $1 \le p < \infty$, is nuclear.

Theorem 4.1 *For $1 \le p < \infty$, let $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ be a nuclear operator and let $T_\tau : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ be a bounded linear operator. Then the pseudo-differential operator $T_\tau T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is a nulcear operator. Moreover, the symbol λ of $T_\tau T_\sigma$ is given by*

$$\lambda(\theta, n) = 4\pi^2 e^{-in\theta} \sum_{k=\infty}^{\infty} u_k(\theta)\widehat{g_k}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z},$$

where

$$u_k(\theta) = \sum_{m=-\infty}^{\infty} e^{im\theta}\tau(\theta, m)\widehat{h_k}(m) = (T_\tau h_k)(\theta), \quad \theta \in [-\pi.\pi].$$

Proof Let $f \in L^p(\mathbb{S}^1)$. Then for all $\theta \in [-\pi.\pi]$,

$$\begin{aligned} &(T_\tau T_\sigma f)(\theta) \\ = &\sum_{m=-\infty}^{\infty} e^{im\theta}\tau(\theta, m)(T_\sigma f)^\wedge(m) \\ = &\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im\theta}\tau(\theta, m)\int_{-\pi}^{\pi} e^{-im\phi}\left(\sum_{n=-\infty}^{\infty} e^{in\phi}\sigma(\phi, n)\hat{f}(n)\right) d\phi. \end{aligned}$$

Since T_σ is nuclear, there exist sequences $\{g_k\}_{k=-\infty}^{\infty}$ in $L^{p'}(\mathbb{S}^1)$ and $\{h_k\}_{k=-\infty}^{\infty}$ in $L^p(\mathbb{S}^1)$ such that

$$\sum_{k=-\infty}^{\infty} \|h_k\|_{L^p(\mathbb{S}^1)}\|g_k\|_{L^{p'}(\mathbb{S}^1)} < \infty$$

and

$$\sigma(\theta, n) = 2\pi e^{-in\theta} \sum_{k=-\infty}^{\infty} h_k(\theta)\widehat{g_k}(-n), \quad (\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}.$$

So,

$$
\begin{aligned}
& (T_\tau T_\sigma f)(\theta) \\
= \; & 2\pi \sum_{m=-\infty}^{\infty} e^{im\theta}\tau(\theta,m) \int_{-\pi}^{\pi} e^{-im\phi} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_k(\phi)\widehat{g_k}(-n)\hat{f}(n)\, d\phi \\
= \; & (4\pi)^2 \sum_{n=-\infty}^{\infty} e^{in\theta} \left[e^{-in\theta} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{im\theta}\tau(\theta,m)\widehat{h_k}(m)\widehat{g_k}(-n) \right] \hat{f}(n)\, d\phi \\
= \; & \sum_{n=-\infty}^{\infty} e^{in\theta}\lambda(\theta,n)\hat{f}(n), \quad \theta \in [-\pi,\pi],
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda(\theta,n) &= 4\pi^2 e^{-in\theta} \sum_{k=-\infty}^{\infty} e^{im\theta}\tau(\theta,m)\widehat{h_k}(m)\widehat{g_k}(-n) \\
&= 4\pi^2 e^{-in\theta} \sum_{k=-\infty}^{\infty} u_k(\theta)\widehat{g_k}(-n), \quad (\theta,n) \in \mathbb{S}^1 \times \mathbb{Z},
\end{aligned}
$$

where

$$
u_k(\theta) = \sum_{k=-\infty}^{\infty} e^{im\theta}\tau(\theta,m)\widehat{h_k}(m), \quad \theta \in [-\pi,\pi].
$$

Since $T_\tau : L^p(\mathbb{S}^1) \to T_\tau(\mathbb{S}^1)$ is a bounded linear operator, it follows that there exists a positive constant C such that

$$
\|u_k\|_{L^p(\mathbb{S}^1)} = \|T_\tau u_k\|_{L^p(\mathbb{S}^1)} \leq C\|h_k\|_{L^p(\mathbb{S}^1)}, \quad k \in \mathbb{Z},
$$

and the proof is complete. □

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References

[1] D. Cardona, Hölder estimates for pseudo-differential operators on $\mathbb{T}^1$, *J. Pseudo-Differ. Oper. Appl.* **5** (2014), 517–525.

[2] J. Delgado, The trace of nuclear operators on $L^p(\mu)$ for σ-finite Borel measureson second countable spaces, *Integr. Equ. Oper. Theory* **68** (2010), 61–74.

[3] J. Delgado, A trace formula for nuclear operatorson L^p, in *Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations*, Operator Theory: Advances and Applications **205**, Birkhäuser, 2010, 181–193.

[4] J. Delgado and M. Ruzhansky, Kernel and symbol criteria for Schatten classes and r-nuclarity on compact manifolds, *C. R. Acad. Sci. Paris, Ser. I* **352** (2014), 779–784.

[5] J. Delgado and M. Ruzhansky, L_p nuclarity, traces, and Grothendieck–Lidskii formula on compact Lie groups, *J. Math. Pures Appl.* **102** (2014), 153–172.

[6] J. Delgado and M. W. Wong, L^p-nuclear pseudo-differential operators on $\mathbb{Z}$ and $\mathbb{S}^1$, Proc. Amer. Math. Soc. **141** (2013), 3935–3942.

[7] M. B. Ghaemi, E. Nabizadeh Morsalfard and M. Jamalpour Birgani, A study on the adjoint of pseudo-differential operators on $\mathbb{S}^1$ and $\mathbb{Z}$, *J. Pseudo-Differ. Oper. Appl.* **6** (2015), 197–203.

[8] M. B. Ghaemi, M. Jamalpour Birgani and E. Nabizadeh Morsalfard, A study on pseudo-differential operators on $\mathbb{S}^1$ and $\mathbb{Z}$, *J. Pseudo-Differ. Oper. Appl.* **7** (2016), 237–247.

[9] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and Determinants of Linear Operators*, Operator Theory: Advances and Applications **116**, Birkhäuser, 2000.

[10] A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucléaires, *Memoirs Amer. Math. Soc.* **16**, 1955.

[11] A. Grothendieck, La theorie de Fredholm, *Bull. Soc. Math. France* **84** (1956), 319–384.

[12] S. Molahajloo, A characterization of compact pseudo-differential operators on $\mathbb{S}^1$, in *Pseudo-Differential Operators: Analysis, Applications*

and Computations, Operator Theory: Advances and Applications **213**, Birkhäuser, 2011, 25–31.

[13] S. Molahajloo and M. W. Wong, Pseudo-differential operators on $\mathbb{S}^1$, in *New Developments in Pseudo-Differential Operators*, Operator Theory: Advances and Applications **189**, Birkhäuser, 2008, 297–306.

[14] S. Molahajloo and M. W. Wong, Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $\mathbb{S}^1$, *J. Pseudo-Differ. Oper. Appl.* **1** (2010), 183–205.

[15] M. Pirhayati, Spectral theory of pseudo-differential operators on $\mathbb{S}^1$, in *Pseudo-Differential Operators: Analysis, Applications and Computations*, Operator Theory: Advances and Applications **213**, Birkhäuser, 2011, 15–25.

[16] M. W. Wong, Trace class Weyl transforms, in *Recent Advances in Operator Theory and its applications*, Operator Theory: Advances and Applications **160**, Birkhäuser, 2005, 469–478.

[17] M. W. Wong, *Discrete Fourier Analysis,* Birkhäuser, 2011.