Analysis of Matrices of Pseudo-Differential Operators with Separable Symbols on $\mathbb{Z}_N$

Qiang Guo and M. W. Wong*
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: pangpangguo@gmail.com,
mwwong@mathstat.yorku.ca

Abstract We give the determinants and upper bounds for the condition numbers of matrices of pseudo-differential operators T_σ on $\mathbb{Z}_N$ with separable symbols σ on $\mathbb{Z}_N \times \mathbb{Z}_N$. For a time series z of size N, we give the computational complexity of $T_\sigma z$. Numerical experiments are also given.

Keywords Fourier transform, Fourier inversion formula, fast Fourier transform, pseudo-differential operator, matrix representation, circulant matrix, Fourier matrix, determinant, condition number, computational complexity

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1 Pseudo-Differential Operators on $\mathbb{Z}_N$

Pseudo-differential operators on $\mathbb{R}^n$ have been well studied, e.g., in [10] among others, and are based on the Fourier inversion formula for the Fourier transform on $\mathbb{R}^n$. An analogous theory is then developed in, for instance, [6, 7] when $\mathbb{R}^n$ is replaced by the unit circle centered at the origin. Pseudo-differential operators on $\mathbb{S}^1$ based on the Fourier inversion formula for Fourier series can be thought of as periodic pseudo-differential operators. Discretizing pseudo-differential operators on $\mathbb{S}^1$ has prompted very recent research into the connections of pseudo-differential operators with modular arithmetic in number theory and finite groups. See, for instance, [3, 4, 5].

Let $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ be the addititive group, where N is a positive integer greater than or equal to 2. and the group law is addition modulo N. Pseudo-differential operators on $\mathbb{Z}_N$ have been studied in [3, 4]. We first recall the notation and the results therein that we need in this paper. A function $z : \mathbb{Z}_N \to \mathbb{C}$ is completely specified by $\begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}$. Thus, we think of the set of all n-tuples with complex entries as functions on $\mathbb{Z}_N$ and we denote it by $L^2(\mathbb{Z}_N)$. It is a finite-dimensional Hilbert space in which the inner product $(\ ,\ )$ and norm $\|\ \|$ are defined by

$$(z, w) = \sum_{n=0}^{N-1} z(n)\overline{w(n)}$$

and

$$\|z\|^2 = (z, z) = \sum_{n=0}^{N-1} |z(n)|^2$$

for all $z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}$ and $w = \begin{pmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{pmatrix}$ in $L^2(\mathbb{Z}_N)$.

An obvious orthonormal basis for $L^2(\mathbb{Z}_N)$ is $S = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$ defined by

$$\epsilon_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad m = 0, 1, \ldots, N-1,$$

where ϵ_m has 1 in the m^{th} position and zeros elsewhere. Another orthonormal basis for $L^2(\mathbb{Z}_N)$ is $\{e_0, e_1, \ldots, e_{N-1}\}$ with

$$e_m = \begin{pmatrix} e_m(0) \\ e_m(1) \\ \vdots \\ e_m(N-1) \end{pmatrix}, \quad m = 0, 1, \ldots, N-1,$$

and

$$e_m(n) = \frac{1}{\sqrt{N}} e^{2\pi i m n/N}, \quad n = 0, 1, \ldots, N-1.$$

The Fourier transform $\hat{z}$ of $z \in L^2(\mathbb{Z}_N)$ is defined by

$$\hat{z} = \begin{pmatrix} \hat{z}(0) \\ \hat{z}(1) \\ \vdots \\ \hat{z}(N-1) \end{pmatrix}$$

with

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n/N}, \quad m = 0, 1, \ldots, N-1.$$

Let $F = \{F_0, F_1, \ldots, F_{N-1}\}$ be the basis for $L^2(\mathbb{Z}_N)$ defined by

$$F_m = \frac{1}{\sqrt{N}} e_m, \quad m = 0, 1, \ldots, N-1.$$

Then for z and w in $L^2(\mathbb{Z}_N)$, we have the Plancherel formula given by

$$(z,w) = \frac{1}{N}(\hat{z},\hat{w})$$

and the Fourier inversion formula to the effect that

$$z = \sum_{m=0}^{N-1} \hat{z}(m) F_m.$$

Let σ be a function on the phase space $\mathbb{Z}_N \times \mathbb{Z}_N$. Then the pseudo-differential operator T_σ on $\mathbb{Z}_N$ is defined by

$$(T_\sigma z)(n) = \sum_{m=0}^{N-1} \sigma(n,m)\hat{z}(m)F_m(n), \quad n = 0,1,\ldots,N-1,$$

for all $z \in L^2(\mathbb{Z}_N)$. The matrix $(T_\sigma)_F$ of T_σ with respect to the basis F given by

$$F = \{F_0, F_1, \ldots, F_{N-1}\}$$

is

$$\begin{aligned}
&(T_\sigma)_F \\
&= \frac{1}{N}\begin{pmatrix}
(\mathcal{F}_1\sigma)(0,0) & (\mathcal{F}_1\sigma)(N-1,0) & \cdots & (\mathcal{F}_1\sigma)(1,0) \\
(\mathcal{F}_1\sigma)(1,1) & (\mathcal{F}_1\sigma)(0,1) & \cdots & (\mathcal{F}_1\sigma)(2,1) \\
\vdots & \vdots & \ddots & \vdots \\
(\mathcal{F}_1\sigma)(N-1,N-1) & (\mathcal{F}_1\sigma)(N-2,N-1) & \cdots & (\mathcal{F}_1\sigma)(0,N-1)
\end{pmatrix} \\
&= \frac{1}{N}((\mathcal{F}_1\sigma)(m-n,n))_{0\le m,n\le N-1},
\end{aligned}$$

where $\mathcal{F}_1\sigma$ denotes the Fourier transform of σ with respect to the first variable. And for separable symbols,

$$\begin{aligned}
&(T_\sigma)_F = \\
&\frac{1}{N}\begin{pmatrix}
\sigma_1(0) & & & \\
& \sigma_1(1) & & \\
& & \ddots & \\
& & & \sigma_1(N-1)
\end{pmatrix}
\begin{pmatrix}
\hat{\sigma}_2(0) & \hat{\sigma}_2(1) & \cdots & \hat{\sigma}_2(N-1) \\
\hat{\sigma}_2(N-1) & \hat{\sigma}_2(0) & \cdots & \hat{\sigma}_2(N-2) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\sigma}_2(1) & \hat{\sigma}_2(2) & \cdots & \hat{\sigma}_2(0)
\end{pmatrix}
\end{aligned}$$

Let $\mathcal{F}_2\sigma$ be the Fourier transform of σ with respect to the second variable. Then the matrix $(T_\sigma)_S$ of the pseudo-differential operator T_σ on $L^2(\mathbb{Z}_N)$ with respect to the basis $S = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$ has the form

$$
\begin{aligned}
&(T_\sigma)_S \\
&= \frac{1}{N}\begin{pmatrix} (\mathcal{F}_2\sigma)(0,0) & (\mathcal{F}_2\sigma)(0,1) & \cdots & (\mathcal{F}_2\sigma)(0,N-1) \\ (\mathcal{F}_2\sigma)(1,N-1) & (\mathcal{F}_2\sigma)(1,0) & \cdots & (\mathcal{F}_2\sigma)(1,N-2) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathcal{F}_2\sigma)(N-1,1) & (\mathcal{F}_2\sigma)(N-1,2) & \cdots & (\mathcal{F}_2\sigma)(N-1,0) \end{pmatrix} \\
&= \frac{1}{N}((\mathcal{F}_2\sigma)(m, n-m))_{0\le m,n\le N-1}.
\end{aligned}
$$

Results on the determinants, condition numbers and computational complexity for matrix representations in terms of the basis S are similar to those in terms of the basis F, so we give the results in this paper using the basis F only.

The aim is to analyze the structure of the matrix of T_σ, where σ is a separable symbol. In Section 2, we give the determinants of the pseudo-differential operators on $\mathbb{Z}_N$ using the basis F. In Section 3, we give upper bounds for the condition numbers. In Section 4, we give the computational complexity, in terms of the bases F, of computing $T_\sigma z$, where $z \in L^2(\mathbb{Z}_N)$. Numerical experiments are given in Section 5.

2 Determinants

Let σ be a separable function on $\mathbb{Z}_N \times \mathbb{Z}_N$, i.e.,

$$\sigma(n,m) = \sigma_1(n)\sigma_2(m), \quad n, m \in \mathbb{Z}_N,$$

where σ_1 and σ_2 are functions on $\mathbb{Z}_N$. Then the matrix $(T_\sigma)_F$ can be decomposed into the product of a diagonal matrix $(T_{\sigma_2})_F$ and a circulant matrix $(T_{\sigma_1})_F$. In fact,

$$(T_\sigma)_F = \frac{1}{N}(T_{\sigma_2})_F(T_{\sigma_1})_F,$$

where

$$(T_{\sigma_2})_F = \begin{pmatrix} \sigma_2(0) & & & \\ & \sigma_2(1) & & \\ & & \ddots & \\ & & & \sigma_2(N-1) \end{pmatrix}$$

and

$$(T_{\sigma_1})_F = \begin{pmatrix} \hat{\sigma}_1(0) & \hat{\sigma}_1(N-1) & \cdots & \hat{\sigma}_1(1) \\ \hat{\sigma}_1(1) & \hat{\sigma}_1(0) & \cdots & \hat{\sigma}_1(2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_1(N-1) & \hat{\sigma}_1(N-2) & \cdots & \hat{\sigma}_1(0) \end{pmatrix}.$$

In fact, the entry in the j^{th} row and k^{th} column of $(T_\sigma)_F$ is

$$\hat{\sigma}_1(j-k), \quad 0 \le j,k \le N-1.$$

The properties of circulant matrices can be found in [2, 9].

In particular, the eigenvalues of $(T_{\sigma_1})_F$ are

$$\sigma_1(0), \sigma_1(1), \dots, \sigma_1(N-1),$$

where multiplicities are taken into account. The eigenvalues of $(T_{\sigma_2})_F$ are

$$\sigma_2(0), \sigma_2(1), \dots, \sigma_2(N-1)$$

if the mutiplicities are counted.

So, we have the following theorem on the determinant of $(T_\sigma)_F$.

Theorem 2.1 *Let σ be a symbol on $\mathbb{Z}_N \times \mathbb{Z}_N$ given by*

$$\sigma(n,m) = \sigma_1(n)\sigma_2(m), \quad n,m \in \mathbb{Z}_N,$$

where σ_1 and σ_2 are functions on $\mathbb{Z}_N$. Then

$$\det{(T_\sigma)_F} = \frac{1}{N^N} \prod_{j=1}^{N-1} \sigma_1(j)\sigma_2(j).$$

3 Condition Numbers

The circulant matrix $(T_{\sigma_1})_F$ can be written as

$$(T_{\sigma_1})_F = \Omega_N^{-1} D_1 \Omega_N,$$

where

$$D_1 = \begin{pmatrix} \sigma_1(0) & & & \\ & \sigma_1(1) & & \\ & & \ddots & \\ & & & \sigma_1(N-1) \end{pmatrix}.$$

and Ω_N is the Fourier matrix define by

$$\begin{aligned} \Omega_N &= \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \omega_N^3 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \omega_N^6 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \omega_N^{3(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix} \\ &= (\omega_N^{mn})_{0 \le m,n \le N-1} \end{aligned}$$

with $\omega_N^{mn} = e^{-2\pi imn/N}$.

Thus, we have the following theorem.

Theorem 3.1 *Let σ be a symbol defined on $\mathbb{Z}_N \times \mathbb{Z}_N$ by*

$$\sigma(n,m) = \sigma_1(n)\sigma_2(m), \quad n, m \in \mathbb{Z}_N.$$

Then

$$(T_\sigma)_F = \frac{1}{N}(T_{\sigma_2})_F \Omega_N^{-1} D_1 \Omega_N.$$

If $\sigma_1(j) \neq 0$ and $\sigma_2(j) \neq 0$ for $j = 0, 1, \ldots N-1$, then the inverse $(T_\sigma)_F^{-1}$ of $(T_\sigma)_F$ has the form

$$(T_\sigma)_F^{-1} = N(T_{\sigma_1})_F^{-1}(T_{\sigma_2})_F^{-1} = N\Omega_N^{-1} D_1 \Omega_N (T_{\sigma_2})_F^{-1} \tag{3.1}$$

with

$$D_1^{-1} = \begin{pmatrix} \frac{1}{\sigma_1(0)} & & & \\ & \frac{1}{\sigma_1(1)} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_1(N-1)} \end{pmatrix}$$

and

$$(T_{\sigma_2})_F^{-1} = \begin{pmatrix} \frac{1}{\sigma_2(0)} & & & \\ & \frac{1}{\sigma_2(1)} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_2(N-1)} \end{pmatrix}.$$

We can now give an upper bound for the condition number $\kappa((T_\sigma)_F)$ of $(T_\sigma)_F$, which is defined by

$$\kappa((T_\sigma)_F) = \|(T_\sigma)_F\| \, \|(T_\sigma)_F^{-1}\|.$$

It should be mentioned that the condition number of a square matrix is at least equal to 1. Detailed accounts of condition numbers can be found in [1, 8].

Theorem 3.2 *Let σ be a symbol defined on $\mathbb{Z}_N \times \mathbb{Z}_N$ by*

$$\sigma(n,m) = \sigma_1(n)\sigma_2(m), \quad n, m \in \mathbb{Z}_N.$$

Then

$$\kappa((T_\sigma)_F) \leq \frac{\max_{0\leq j\leq N-1} |\sigma_1(j)| \, \max_{0\leq j\leq N-1} |\sigma_2(j)|}{\min_{0\leq j\leq N-1} |\sigma_1(j)| \, \min_{0\leq j\leq N-1} |\sigma_2(j)|}.$$

Proof By Theorem 3.1 and (3.1), we get

$$\kappa((T_\sigma)_F) = \|(T_{\sigma_2})_F\| \, \|(T_{\sigma_2})_F^{-1}\| \, \|\Omega_N^{-1}\|^2 \, \|D_1\| \, \|D_1^{-1}\| \, \|\Omega_N\|^2.$$

Since $\frac{1}{\sqrt{N}}\Omega_N$ is unitary, $\Omega_N^{-1} = \frac{1}{N}\overline{\Omega_N}$ and the norm of a diagonal matrix with nonzero entries is given by the maximum of the absolute values of the entries, the theorem is proved. $\square$

4 Computational Complexity

By Theorem 3.1, the computational complexity, i.e., the number of multiplications required to compute $T_\sigma z$, where $z \in L^2(\mathbb{Z}_N)$, is

$$O(N) + O(N\log N) + O(N) + O(N\log N) = O(N\log N).$$

Remark 4.1 Some comments on the computational complexity are in order. Let σ be a function on the phase space $\mathbb{Z}_N \times \mathbb{Z}_N$. Then T_σ, the pseudo-differential operator on $\mathbb{Z}_N$ corresponding to the symbol σ, is defined by

$$(T_\sigma z)(n) = \sum_{m=0}^{N-1} \sigma(n,m)\hat{z}(m)F_m(n)$$

for all $z \in L^2(\mathbb{Z}_N)$, where

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n)e^{-2\pi imn/N}, \quad m = 0,1,\ldots,N-1.$$

For a time series z of size N, the computational complexity of FFT is $O(N\log N)$ converting

$$z(n), \quad n = 0,1,\ldots,N-1,$$

to

$$\hat{z}(m), \quad m = 0,1,\ldots,N-1.$$

After we obtain the Fourier transformed time series, N^2 summations and $2N^2$ multiplications are required, so the computational complexity in general is

$$O(N^2) + O(N\log N) = O(N^2).$$

But if the symbol is separable, then the computational complexity turns out to be just like that of the FFT, i.e., $O(N\log N)$.

5 Numerical Experiments

In this part, we first give the action of pseudo-differential operators on $\mathbb{Z}_N$ on chirp signals. Then three different pseudo-differential operators on $\mathbb{Z}_N$ are studied based on the signal composed of three frequencies.

Example 5.1 We consider a chirp signal with linear instantaneous frequency deviation. The chirp has 1024 samples in 1 second. The instantaneous frequency is 0 at $t = 0$ and crosses 64 Hz at $t = 1$ second. The chirp signal and the transformed ones with three sybmols

$$\sigma(n, m) = n/N,$$

$$\sigma(n, m) = m/N$$

and $\sigma(n, m) = nm/N^2$ are shown in Figure 1.

If the symbol σ is only dependent on time n, where $\sigma(n, m) = \frac{n}{N}$, then

$$(T_\sigma z)(n) = \frac{n}{N} \sum_{m=0}^{N-1} \hat{z}(m) F_m(n) = \frac{n}{N} z(n),$$

which diminishes the original time series and is confirmed in Figure 1. If σ is only dependent on frequency m with $\sigma(n, m) = \frac{m}{N}$, the bottom left plot shows that the amplitude of the signal is increasing with frequency. And the bottom right plot shows the combined effects of the first two symbols.

Example 5.2 In this example, three symbols are tested. They are

$$\sigma(n, m) = \sin\left(\frac{\pi n}{N}\right),$$

$$\sigma(n, m) = 0.02 e^{5m/N}$$

and

$$\sigma(n, m) = \frac{1}{(mn/N^2) + 1}$$

for all n and m in $\mathbb{Z}_N$. We consider the following signal having length 256 with

$$z(n) = \begin{cases} \cos\left(2\pi \frac{4}{128} n\right), & n < \frac{N}{8} \text{ or } \frac{N}{4} < n \leq \frac{N}{2}, \\ \cos\left(2\pi \frac{16}{128} n\right), & n > \frac{N}{2}, \\ \cos\left(2\pi \frac{4}{128} n\right) + \cos\left(2\pi \frac{64}{128} n\right), & \frac{N}{8} \leq n \leq \frac{N}{4}, \end{cases}$$

which is composed of low frequency for the first half time series, medium frequency for the latter half, and high frequency inputs added when

$$\frac{N}{8} \leq n \leq \frac{N}{4}.$$

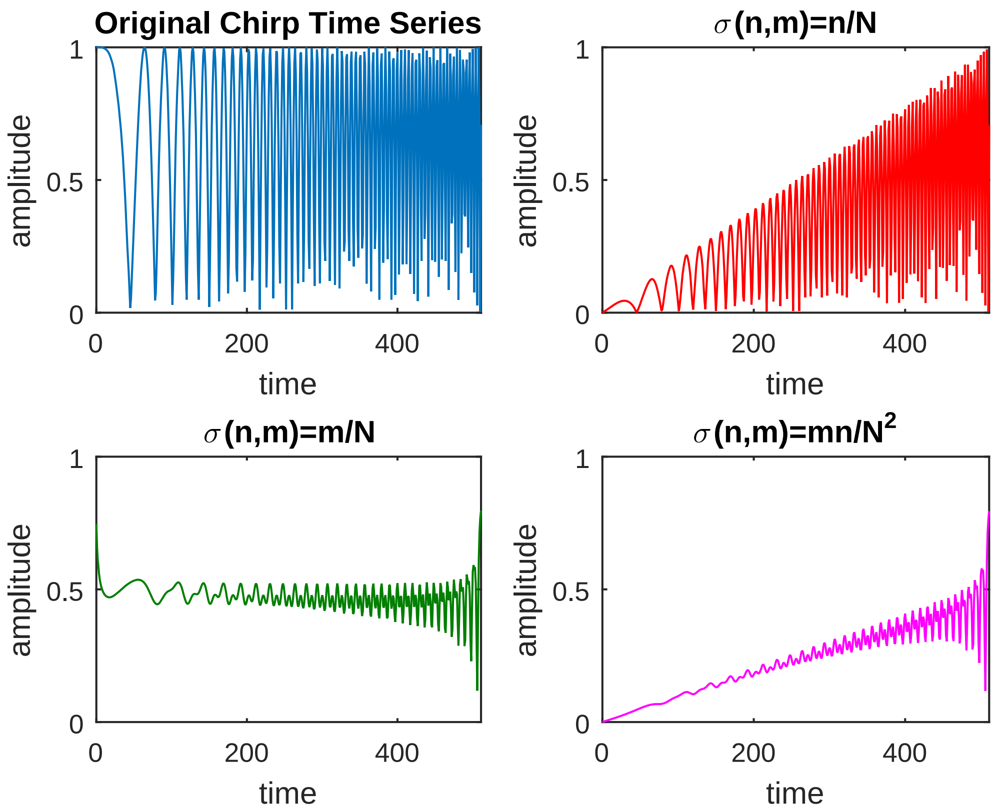

Figure 1: Chirp Time Series

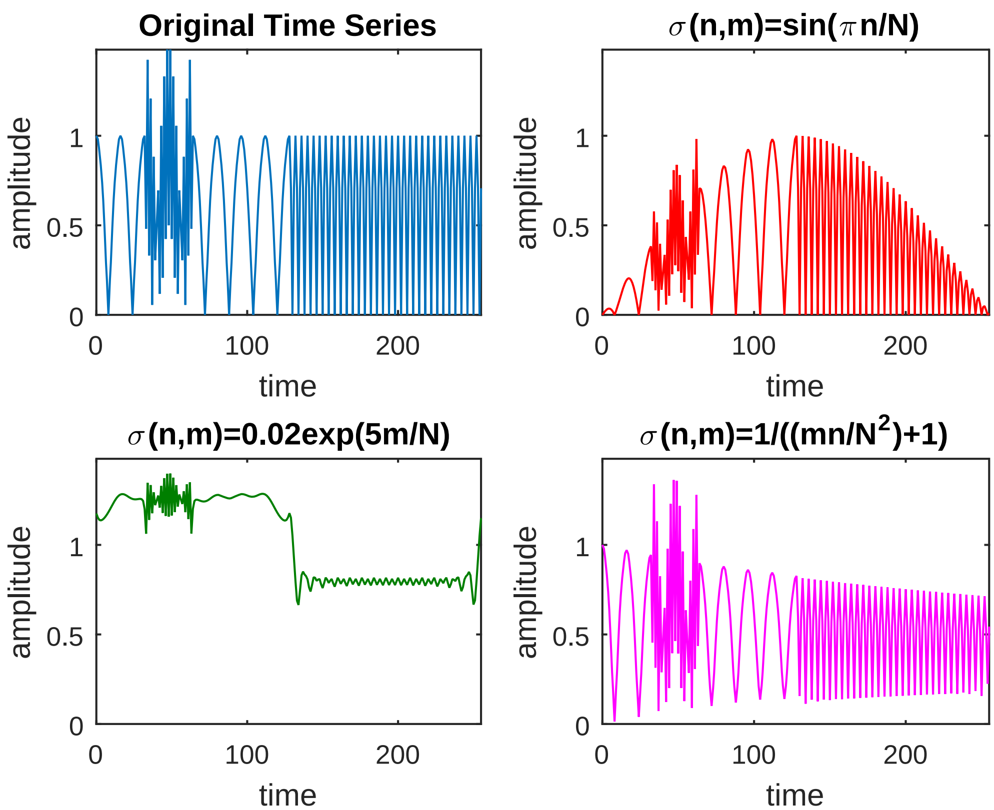

Figure 2: Periodic Time Series

Figure 2 shows the original time series and the transformed ones with different symbols. The symbol $\sigma = \sin(\frac{n}{N})$ imposes a sine function along time direction in the upper right plot, the symbol $\sigma = 0.02e^{5m/N}$ has its impact only on the frequency in the bottom left plot, which separates the high and low frequency parts. And the bottom right plot shows the mixed impact on time and frequency with $\sigma = \frac{1}{(mn/N^2)+1}$.

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