Solutions to Test 2

1. Let $\varphi \in \mathcal{S}$. Then

$$\begin{aligned}(\partial H)(\varphi) &= -H(\partial\varphi) \\ &= -\int_{-\infty}^{\infty} H(x)(\partial\varphi)(x)\,dx \\ &= -\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx \\ &= -\int_{0}^{\infty} \varphi'(x)\,dx.\end{aligned}$$

Using the fundamental theorem of calculus, we have

$$(\partial H)(\varphi) = -\varphi(x)|_0^\infty = \varphi(0) = \delta(\varphi).$$

Therefore

$$\partial H = \delta.$$

2. We first note that

$$f(x) = \frac{1}{2}(1-\cos(2x)) = \frac{1}{2} - \frac{1}{4}e^{i2x} - \frac{1}{4}e^{i(-2)x}, \quad x \in \mathbb{R}.$$

Therefore

$$\hat{f} = (2\pi)^{1/2}\left(\frac{1}{2}\delta - \frac{1}{4}\delta_{-2} - \frac{1}{4}\delta_2\right).$$

3. See 6.3 in Assignment 4.

4. Since δ is a tempered distribution on $\mathbb{R}^n$, we have

$$u(\cdot,t) = \delta * k_t = k_t, \quad t > 0.$$

Therefore a solution u of the initial value problem is given by

$$u(x,t) = k_t(x), \quad x \in \mathbb{R}^n, t > 0.$$