Answers to Assignment 4

6.1. Taking the partial Fourier transform of u with respect to x, we get

$$\begin{cases} \frac{\partial \hat{u}}{\partial t}(\xi,t) = -|\xi|^2\hat{u}(\xi,t) + a\hat{u}(\xi,t), & \xi \in \mathbb{R}^n, t > 0, \\ \hat{u}(\xi,t) = \hat{f}(\xi), & \xi \in \mathbb{R}^n. \end{cases}$$

So,

$$\hat{u}(\xi,t) = C(\xi)e^{-(|\xi|^2-a)t}, \quad \xi \in \mathbb{R}^n, t > 0.$$

Putting $t = 0$, we get

$$C(\xi) = \hat{f}(\xi), \quad \xi \in \mathbb{R}^n.$$

Therefore

$$\hat{u}(\xi,t) = e^{at}\hat{f}(\xi)e^{-|\xi|^2 t}, \quad \xi \in \mathbb{R}^n, t > 0.$$

Hence

$$u(x,t) = e^{at}(k_t * f)(x) = \frac{e^{at}}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)}f(y)\,dy, \quad x \in \mathbb{R}^n, t > 0.$$

6.2. We begin with the heat kernel

$$k_t(x) = \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)}, \quad x \in \mathbb{R}^n, t > 0.$$

Then

$$\begin{aligned} \frac{\partial k_t}{\partial t} &= \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)}\frac{|x|^2}{4t^2} \\ &\quad + \frac{1}{(4\pi)^{n/2}}(-n/2)t^{-(n/2)-1}e^{-|x|^2/(4t)} \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $t > 0$. Next, for $j = 1, 2, \ldots, n$,

$$\frac{\partial k_t}{\partial x_j} = \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)}(-x_j/(2t)), \quad x \in \mathbb{R}^n, t > 0,$$

and hence

$$\begin{aligned} \frac{\partial^2 k_t}{\partial x_j^2} &= \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}(-1/(2t) \\ &\quad + \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}(x_j^2/(4t^2)) \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $t > 0$. So,

$$\begin{aligned} \Delta k_t &= \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}(-n/(2t)) \\ &\quad + \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}(|x|^2/(4t^2)) \end{aligned}$$

Therefore

$$\frac{\partial k_t}{\partial t} = \Delta k_t.$$

Finally, As in page 38, write $T = D^\alpha f$, where α is a multi-index and f is a continuous tempered function on $\mathbb{R}^n$. Then

$$(T * k_t)(x) = (D^\alpha f)(\varphi) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(y)(D^\alpha k_t)(x-y)\, dy$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Therefore

$$\begin{aligned} (\partial_t(T * k_t))(x) &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(y)(D^\alpha \partial_t k_t)(x-y)\, dy \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(y)(D^\alpha \Delta k_t)(x-y)\, dy \\ &= (\Delta(T * k_t))(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $t > 0$.

6.3. **(Count this question in two parts)** Since for every positive time t, $k_t \in \mathcal{S}$, we can prove a more general result that the convolution $T * \varphi$ of a tempered distribution T with a Schwartz function φ is a tempered function in $C^\infty(\mathbb{R}^n)$.

(i) (Tempered Function on $\mathbb{R}^n$) See page 38.

(ii) (Function in $C^\infty(\mathbb{R}^n)$) We again write $T = D^\alpha f$ where α is a multi-index and f is a continuous tempered function on $\mathbb{R}^n$. Then

$$(T * \varphi)(x) = (D^\alpha f)((\tilde{\varphi})_{-x}) = \int_{\mathbb{R}^n} f(y)(D^\alpha \varphi)(x-y)\,dy$$

fprall $x \in \mathbb{R}^n$. For every multi-index β,

$$(D^\beta(T * \varphi))(x) = (-1)^{|\beta|} \int_{\mathbb{R}^n} f(y)(D^{\alpha+\beta}\varphi)(x-y)\,dy$$

for all $x \in \mathbb{R}^n$. Therefore

$$(D^\beta(T * \varphi))(x) = (-1)^{|\alpha|}(D^{\alpha+\beta} f)(\varphi) \in \mathbb{C}$$

for all $x \in \mathbb{R}^n$. So, $T * \varphi \in C^\infty(\mathbb{R}^n)$.

6.5. A solution u is given by

$$u(x,t) = f(x), \quad x \in \mathbb{R}^n, t > 0.$$

Indeed,

$$\frac{\partial u}{\partial t}(x,t) = 0, \quad x \in \mathbb{R}^n, t > 0,$$

and

$$(\Delta u)(x,t) = (\Delta f)(x) = 0, \quad x \in \mathbb{R}^n, t > 0.$$

Therefore

$$\frac{\partial u}{\partial t}(x,t) = (\Delta u)(x,t), \quad x \in \mathbb{R}^n, t > 0.$$

Finally

$$u(x,0) = f(x), \quad x \in \mathbb{R}^n.$$

6.6. The answer is no. Let

$$f(x) = 1, \quad x \in \mathbb{R}^n.$$

Then $f \in L^\infty(\mathbb{R}^n)$. But

$$\begin{aligned} u(x,t) &= (k_t * f)(x) \\ &= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)} dy \\ &= \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-|y|^2/(4t)} dy = 1. \end{aligned}$$