Answers to Assignment 3

5.2. For all functions $\varphi, \psi \in \mathcal{S}$ and all complex numbers $a, b \in \mathbb{C}$,

$$\begin{aligned}(\partial^\alpha T)(a\varphi + b\psi) &= (-1)^{|\alpha|}T(\partial^\alpha(a\varphi + b\psi)) \\ &= (-1)^{|\alpha|}T(a\partial^\alpha\varphi + b\partial^\alpha\psi) \\ &= a(-1)^{|\alpha|}T(\partial^\alpha\varphi) + b(-1)^{|\alpha|}T(\partial^\alpha\psi) \\ &= a(\partial^\alpha T)(\varphi) + b(\partial^\alpha T)(\psi).\end{aligned}$$

Therefore $\partial^\alpha T$ is a linear functional on $\mathcal{S}$. Let $\{\varphi\}_{j=1}^\infty$ be a sequence in $\mathcal{S}$ such that $\varphi_j \to 0$ in $\mathcal{S}$ as $j \to \infty$. Then

$$(\partial^\alpha T)(\varphi_j) = (-1)^{|\alpha|}T(\partial^\alpha\varphi_j) \to 0$$

as $j \to \infty$ because

$$\partial^\alpha\varphi_j \to 0$$

in $\mathcal{S}$ as $j \to \infty$ and T is a tempered distribution.

5.3 For all $\varphi \in \mathcal{S}$,

$$\begin{aligned}(D^\alpha\delta)^\wedge(\varphi) &= (D^\alpha\delta)(\hat{\varphi}) \\ &= (-1)^{|\alpha|}\delta(D^\alpha\hat{\varphi}) \\ &= (-1)^{|\alpha|}\delta(((-x)^\alpha\varphi)^\wedge) \\ &= (-1)^{|\alpha|}(-x^\alpha\varphi)^\wedge(0) \\ &= (-1)^{|\alpha|}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot 0}(-x)^\alpha\varphi(x)\,dx \\ &= (-1)^{|\alpha|}(2\pi)^{-n/2}\int_{\mathbb{R}^n} (-x)^\alpha\varphi(x)\,dx \\ &= (2\pi)^{-n/2}x^\alpha(\varphi).\end{aligned}$$

Therefore $(D^\alpha\delta)^\wedge = (2\pi)^{-n/2}x^\alpha$.

5.5. For all $\varphi \in \mathcal{S}$,

$$\begin{aligned}
\hat{f}(\varphi) &= f(\hat{\varphi}) \\
&= \int_{\mathbb{R}^n} \xi^\alpha \hat{\varphi}(\xi)\, d\xi \\
&= \int_{\mathbb{R}^n} (D^\alpha \varphi)^\wedge(\xi)\, d\xi \\
&= (2\pi)^{n/2}(D^\alpha\varphi)(0) \\
&= (2\pi)^{n/2}\delta(D^\alpha\varphi) \\
&= (2\pi)^{n/2}(-1)^{|\alpha|}(D^\alpha\delta)(\varphi).
\end{aligned}$$

Therefore

$$\hat{f} = (2\pi)^{n/2}(-1)^{|\alpha|}D^\alpha\delta.$$

5.6. We begin with (c). For all $\varphi \in \mathcal{S}$,

$$\begin{aligned}
\hat{f}(\varphi) &= f(\hat{\varphi}) \\
&= \int_{-\infty}^{\infty} e^{iax}\hat{\varphi}(x)\, dx \\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iax}\left(\int_{-\infty}^{\infty} e^{-ixy}\varphi(y)\, dy\right) dx \\
&= \int_{-\infty}^{\infty} \varphi(y)\left((2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{i(a-y)x}\, dx\right) dy \\
&= (2\pi)^{1/2}\int_{-\infty}^{\infty} \delta(a-y)\varphi(y)\, dy \\
&= (2\pi)^{1/2}\varphi(a) \\
&= (2\pi)^{1/2}\varphi_a(0) \\
&= (2\pi)^{1/2}\delta(\varphi_a) \\
&= (2\pi)^{1/2}\delta_{-a}(\varphi),
\end{aligned}$$

where δ_{-a} is the translation of δ defined by

$$\delta_{-a}(\varphi) = \varphi(a), \quad \varphi \in \mathcal{S}.$$

Therefore

$$\hat{f} = (2\pi)^{1/2}\delta_{-a}.$$

(a) Since

$$\cos\, x = \frac{e^{ix} + e^{-ix}}{2},$$

it follows from (c) that

$$\hat{f} = (2\pi)^{1/2}\frac{\delta_{-1} + \delta_1}{2} = \sqrt{\frac{\pi}{2}}(\delta_{-1} + \delta_1).$$

(b) Since

$$\sin\, x = \frac{e^{ix} - e^{-ix}}{2i},$$

we get

$$\hat{f} = \frac{1}{i}\sqrt{\frac{\pi}{2}}(\delta_{-1} - \delta_1).$$

5.11. For all $\varphi \in \mathcal{S}$,

$$\begin{aligned} (D^\alpha T)^\wedge(\varphi) &= (D^\alpha T)(\hat{\varphi}) \\ &= (-1)^{|\alpha|}T(D^\alpha \hat{\varphi}) \\ &= (-1)^{|\alpha|}T(((-x)^\alpha \varphi)^\wedge) \\ &= (-1)^{|\alpha|}\hat{T}((-x)^\alpha \varphi) \\ &= (x^\alpha \hat{T})(\varphi). \end{aligned}$$

Therefore

$$(D^\alpha T)^\wedge = x^\alpha \hat{T}.$$

5.12. We use the regularity theorem on page 38 to the effect that $T = D^\alpha f$ for some multi-index α and some continuous tempered function f on $\mathbb{R}^n$.

Then for all $\psi \in \mathcal{S}$,

$$
\begin{aligned}
(T * \varphi)^\wedge(\psi) &= (T * \varphi)(\hat{\psi}) \\
&= (D^\alpha f) * \varphi)(\hat{\psi}) \\
&= (D^\alpha f)((\tilde{\varphi})_{-x}))(\hat{\psi}) \\
&= \int_{\mathbb{R}^n} ((D^\alpha f)((\tilde{\varphi})_{-x}))\hat{\psi}(x)\, dx \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) D_y^\alpha (\varphi(x-y))\, dy \right) \hat{\psi}(x)\, dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y)(D^\alpha \varphi)(x-y)\, dy \right) \hat{\psi}(x)\, dx \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} (D^\alpha \varphi)(x-y)\hat{\psi}(x)\, dx \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} (T_{-y}(D^\alpha \varphi))(x)\hat{\psi}(x)\, dx \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} (M_{-y}(D^\alpha \varphi)^\wedge)(\xi)\psi(\xi)\, d\xi \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-iy\cdot\xi} \xi^\alpha \hat{\varphi}(\xi)\psi(\xi)\, d\xi \right) dy \\
&= \int_{\mathbb{R}^n} \xi^\alpha \hat{\varphi}(\xi)\psi(\xi) \left(\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\, dy \right) d\xi \\
&= (2\pi)^{n/2} \int_{\mathbb{R}^n} \xi^\alpha \hat{f}(\xi)\hat{\varphi}(\xi)\psi(\xi)\, d\xi \\
&= (2\pi)^{n/2} \int_{\mathbb{R}^n} (D^\alpha f)^\wedge(\xi)\hat{\varphi}(\xi)\psi(\xi)\, d\xi \\
&= (2\pi)^{n/2} (\hat{\varphi}(D^\alpha f)^\wedge)(\psi) \\
&= ((2\pi)^{n/2}\hat{\varphi}\hat{T})(\psi).
\end{aligned}
$$

Therefore

$$(T * \varphi)^\wedge = (2\pi)^{n/2}\hat{\varphi}\hat{T}.$$