Answers to Assignment 2

1.1. For all $x \in \mathbb{R}^n$ and all multi-indices α,

$$|x^\alpha| = |x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}|.$$

For $j = 1, 2, \ldots, n$,

$$|x_j| \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |x|.$$

Therefore

$$|x^\alpha| \leq |x|^{\alpha_1} |x|^{\alpha_2} \cdots |x|^{\alpha_n} = |x|^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} = |x|^{|\alpha|}.$$

1.2. We obtain from first-year calculus that for all nonnegative integers m and n,

$$\left(\frac{d}{dx}\right)^m (x^n) = \begin{cases} \binom{n}{m} m! x^{n-m}, & m \leq n, \\ 0, & m > n. \end{cases}$$

Therefore for all multi-indices α and β with $\beta \leq \alpha$,

$$\begin{aligned} \partial^\beta x^\alpha &= \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n} (x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \\ &= (\partial_1^{\beta_1} x_1^{\alpha_1})(\partial_2^{\beta_2} x^{\alpha_2}) \cdots (\partial_n^{\beta_n} x_n^{\alpha_n}) \\ &= \prod_{j=1}^{n} \partial_j^{\beta_j} x_j^{\alpha_j}. \end{aligned}$$

For $j = 1, 2, \ldots, n$,

$$\partial_j^{\beta_j} (x_j^{\alpha_j}) = \binom{\alpha_j}{\beta_j} \beta_j x_j^{\alpha_j - \beta_j}.$$

Therefore

$$\partial^\beta x^\alpha = \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j} \beta_j! x^{\alpha_j - \beta_j} = \binom{\alpha}{\beta} \beta! x^{\alpha - \beta}.$$

If β is not $\leq \alpha$, then there exists a positive integer j with $1 \leq j \leq n$ and $\beta_j > \alpha_j$. So,

$$\partial_j^{\beta_j} x^{\alpha_j} = 0,$$

which gives

$$\partial^\beta x^\alpha = 0.$$

3.5.(i) Let β and γ be multi-indices. Then by Proposition 1.1,

$$x^\gamma(\partial^\beta(x^\alpha\varphi))(x) = x^\gamma \sum_{\delta\le\beta}\binom{\beta}{\delta}(\partial^\delta x^\alpha)(\partial^{\beta-\delta}\varphi)(x), \quad x\in\mathbb{R}^n.$$

By Exercise 1.2,

$$\partial^\delta x^\alpha = \begin{cases} \binom{\alpha}{\delta}\delta! x^{\alpha-\delta}, & \delta\le\alpha, \\ 0, & \delta>\alpha. \end{cases}$$

So, we can assume that $\delta\le\alpha$ and we obtain

$$\begin{aligned} & x^\gamma(\partial^\beta(x^\alpha\varphi))(x) \\ = \;& x^\gamma \sum_{\delta\le\beta}\binom{\beta}{\delta}\binom{\alpha}{\delta}\delta! x^{\alpha-\delta}(\partial^{\beta-\delta}\varphi)(x) \\ = \;& \sum_{\delta\le\beta}\binom{\beta}{\delta}\binom{\alpha}{\delta}\delta! x^{\alpha+\gamma-\delta}(\partial^{\beta-\delta}\varphi)(x) \end{aligned}$$

for all $x\in\mathbb{R}^n$. Since $\varphi\in\mathcal{S}$, it follows that there exists a positive constant $C_{\alpha,\beta,\gamma,\delta}$ such that

$$|x^{\alpha+\gamma-\delta}(\partial^{\beta-\delta}\varphi)(x)| \le C_{\alpha,\beta,\gamma,\delta}$$

for all $x\in\mathbb{R}^n$. Therefore

$$|x^\gamma(\partial^\beta(x^\alpha\varphi))(x)| \le \sum_{\delta\le\beta}\binom{\beta}{\delta}\binom{\alpha}{\delta}\delta! C_{\alpha,\beta,\gamma,\delta}, \quad x\in\mathbb{R}^n.$$

Therefore

$$\sup_{x\in\mathbb{R}^n}|x^\gamma(\partial^\beta(x^\alpha\varphi))(x)| < \infty.$$

This proves that $x^\alpha\varphi\in\mathcal{S}$.

(ii) Let γ and β be multi-indices. Then

$$x^\gamma(\partial^\beta(\partial^\alpha\varphi))(x) = x^\gamma(\partial^{\alpha+\beta}\varphi)(x), \quad x\in\mathbb{R}^n.$$

Since $\varphi \in \mathcal{S}$, we can find a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|x^\gamma(\partial^{\alpha+\beta}\varphi)(x)| \leq C_{\alpha,\beta,\gamma}, \quad x \in \mathbb{R}^n.$$

Therefore

$$\sup_{x\in\mathbb{R}^n} |x^\gamma(\partial^\beta(\partial^\alpha\varphi))(x)| < \infty.$$

So, $\partial^\alpha\varphi \in \mathcal{S}$.

3.6. We begin with the fact that

$$(**) \quad f, g \in \mathcal{S} \Rightarrow fg \in \mathcal{S}.$$

Indeed, for all multi-indices α and β, we obtain grom Proposition 1.1 that

$$\begin{aligned} x^\alpha(\partial^\beta(fg))(x) &= x^\alpha \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} (\partial^\gamma f)(x)(\partial^{\beta-\gamma}g)(x) \\ &= \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} x^\alpha(\partial^\gamma f)(x)(\partial^{\beta-\gamma}g)(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$. Since $f, g \in \mathcal{S}$, we can find positive constants $C_{\alpha,\gamma}$ and $C_{\beta,\gamma}$ such that

$$|x^\alpha(\partial^\gamma f)(x)| \leq C_{\alpha,\gamma}, \quad x \in \mathbb{R}^n,$$

and

$$|(\partial^{\beta-\gamma}g)(x)| \leq C_{\beta,\gamma}, \quad x \in \mathbb{R}^n.$$

So,

$$|x^\alpha(\partial^\beta(fg))(x)| \leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} C_{\alpha,\gamma}C_{\beta,\gamma}, \quad x \in \mathbb{R}^n.$$

Therefore

$$\sup_{x\in\mathbb{R}^n} |x^\alpha(\partial^\beta(fg))(x)| < \infty,$$

which is the same as saying that $fg \in \mathcal{S}$. By Theorem 4.1 and $(**)$,

$$(\varphi * \psi)^\wedge = (2\pi)^{n/2}\hat{\varphi}\hat{\psi} \in \mathcal{S}.$$

By the Fourier Inversion Theorem (Theorem 4.8),

$$\varphi * \psi = ((\varphi * \psi)^\wedge)^\vee \in \mathcal{S}.$$

4.1. For all $\xi \in \mathbb{R}^n$,

$$\begin{aligned} |\hat{f}(\xi)| &= \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx\right| \\ &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(x)|\,dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\,dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot 0} f(x)\,dx = \hat{f}(0). \end{aligned}$$

4.2. By the Fourier Inversion Formula,

$$\begin{aligned} (\varphi * \varphi)(x) &= ((\varphi * \varphi)^\wedge)^\vee(x) \\ &= (2\pi)^{n/2} (\hat{\varphi}\hat{\varphi})^\vee(x) \\ &= (2\pi)^{n/2} (\varphi^2)^\vee(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$. But for all $x \in \mathbb{R}^n$,

$$\begin{aligned} (\varphi^2)^\vee(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi^2(\xi)\,d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-|\xi|^2} d\xi. \end{aligned}$$

Let $\xi = \frac{\eta}{\sqrt{2}}$. Then $d\xi = 2^{-n/2} d\eta$ and for all $x \in \mathbb{R}^n$,

$$(\varphi^2)^\vee(x) = (2\pi)^{-n/2} 2^{-n/2} \int_{\mathbb{R}^n} e^{i(x/\sqrt{2})\cdot\eta} e^{-|\eta|^2/2} d\eta = 2^{-n/2} e^{-|x|^2/4}.$$

Thus, for all $x \in \mathbb{R}^n$,

$$(\varphi * \varphi)(x) = (2\pi)^{n/2} 2^{-n/2} e^{-|x|^2/4} = \pi^{n/2} e^{-|x|^2/4}.$$

4.3. We first note that for $j = 1, 2, \ldots, n$, $D_j = -i\partial_j$ and hence

$$\frac{\partial}{\partial x_j} = \partial_j = iD_j.$$

Therefore

$$\Delta = \frac{\partial^2}{\partial x_j^2} = \partial_j^2 = -D_j^2.$$

By the first part of Proposition 4.2, we get for all $\varphi \in \mathcal{S}$,

$$\widehat{\Delta\varphi}(\xi) = -\sum_{j=1}^{n} \widehat{D_j^2\varphi}(\xi) = -\sum_{j=1}^{n} \xi^2 \hat{\varphi}(\xi) = -|\xi|^2 \hat{\varphi}(\xi)$$

for all $\xi \in \mathbb{R}^n$.