Answers to Assignment 1

3.1. We have

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int_{\mathbb{R}^n} f(x-y)\,dy.$$

Let $z = x - y$. Then

$$(f * g)(x) = \int_{\mathbb{R}^n} f(z)\,dz$$

for all $x \in \mathbb{R}^n$. (The answer is a number.)

3.2. We begin with

$$\int_{\mathbb{R}^n} (f * g)(x)\,dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)g(y)\,dy \right) dx.$$

Interchanging the order of integration, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (f * g)(x)\,dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)g(y)\,dx \right) dy \\ &= \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} f(x-y)\,dx \right) dy \end{aligned}$$

Let $z = x - y$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} (f * g)(x)\,dx &= \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} f(z)\,dz \right) dy \\ &= \left(\int_{\mathbb{R}^n} f(z)\,dz \right) \left(\int_{\mathbb{R}^n} g(y)\,dy \right) \\ &= \left(\int_{\mathbb{R}^n} f(x)\,dx \right) \left(\int_{\mathbb{R}^n} g(x)\,dx \right). \end{aligned}$$

3.3. (a) By Hölder's inequality,

$$|(f * g)(x)| = \left| \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \right| = \int_{\mathbb{R}^n} |f(x-y)g(y)|\,dy \leq \|f\|_p \|g\|_{p'}$$

for all $x \in \mathbb{R}^n$.

3.3 (b) We need to prove that

$$(f * g)(x + h) \to (f * g)(x)$$

as $h \to 0$ for all $x \in \mathbb{R}^n$. But

$$\begin{aligned} & |(f * g)(x + h) - (f * g)(x)| \\ = & \left| \int_{\mathbb{R}^n} f(x + h - y) g(y)\, dy - \int_{\mathbb{R}^n} f(x - y) g(y)\, dy \right| \\ = & \left| \int_{\mathbb{R}^n} (f(x + h - y) - f(x - y)) g(y)\, dy \right| \\ \leq & \int_{\mathbb{R}^n} |f(x + h - y) - f(x - y)|\, |g(y)|\, dy \\ = & \int_{\mathbb{R}^n} |f_{x+h}(-y) - f_x(-y)|\, |g(y)|\, dy \end{aligned}$$

for all x and h in $\mathbb{R}^n$. By Hölder's inequality, we get for all x and h in $\mathbb{R}^n$,

$$|(f * g)(x + h) - f(x)| \leq \|f_{x+h} - f_x\|_p \|g\|_{p'}.$$

So, by the L^p-continuity of translation,

$$|(f * g)(x + h) - (f * g)(x)| \to 0$$

as $h \to 0$. Therefore $f * g$ is continuous at x. But x is an arbitrary point in $\mathbb{R}^n$. So, $f * g$ is continuous on $\mathbb{R}^n$.

4. Let φ be the function on $\mathbb{R}^n$ defined by

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Then $\varphi \in C_0^\infty(\mathbb{R}^n)$. For every positive number ε, we define the function φ_ϵ on $\mathbb{R}^n$ by

$$\varphi_\varepsilon(x) = e\varphi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n.$$

Then φ_ϵ satisfies the three required conditions. (Make sure that you check these.)