Mathematics 1300 3.0 AF
Solutions to Test 1

1. Solution: (i) Let $x \to 0$ along the numbers $\frac{1}{\left(n+\frac{1}{2}\right)\pi}$. Then

$$\sin\frac{1}{x} = \sin\left(n+\frac{1}{2}\right)\pi = \sin n\pi\cos\frac{\pi}{2} + \cos n\pi\sin\frac{\pi}{2} = (-1)^n.$$

So, $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

Solution: (ii) Since

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

and $\lim_{x\to 0} x^2 = 0$, it follows from the squeeze theorem that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

2. Solution: $|\sqrt{x-1}| < \varepsilon$ if $0 < x-1 < \varepsilon^2$ or if $1 < x < 1+\varepsilon^2$. So, let δ be any positive number $\le \varepsilon^2$. Then

$$1 < x < 1+\delta \Rightarrow |f(x)| = \sqrt{x-1} < \sqrt{\delta} \le \varepsilon.$$

3. Solution: Let $f(x) = \cos x - x$. Then $f(0) = 1$ and $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$ Therefore 0 is between $f(0)$ and $f\left(\frac{\pi}{2}\right)$ By the Intermediate Value Theorem, there exists a number c in $\left(0, \frac{\pi}{2}\right)$ such that

$$f(c) = 0.$$

Therefore

$$\cos c = c.$$

4. Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

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\begin{aligned}
&= \lim_{h\to 0} \frac{\sqrt{x+h-1}-\sqrt{x-1}}{h} \frac{\sqrt{x+h-1}+\sqrt{x-1}}{\sqrt{x+h-1}+\sqrt{x-1}} \\
&= \lim_{h\to 0} \frac{(x+h-1)-(x-1)}{h(\sqrt{x+h-1}+\sqrt{x-1}} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h-1}+\sqrt{x-1}} \\
&= \frac{1}{2\sqrt{x-1}}.
\end{aligned}
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The domain of f is $[1,\infty)$ and the domain of f' is $(1,\infty)$.