Continuous Inversion Formulas for Multi-Dimensional Modified Stockwell Transforms[1]

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Abstract We introduce multi-dimensional modified Stockwell transforms and give for them continuous inversion formulas, extending the work in [1] in different directions.

Key Words Fourier transforms, Gabor transforms, wavelet transforms, Stockwell transforms, modified Stockwell transforms, continuous inversion formulas

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1 Introduction

Let f be a signal in $L^2(\mathbb{R}^n)$. Then we can extract the frequency content of f using the Fourier transform $\mathcal{F}$ given by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

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for all $\xi \in \mathbb{R}^n$. The Fourier transform cannot provide any information on the time/space localization of the frequencies. There are many ways of getting time-frequency information from a signal. For example, we can use the Gabor transform G defined by

$$(Gf)(b,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) e^{-|x-b|^2/2} dx, \quad b,\xi \in \mathbb{R}^n. \tag{1.1}$$

Instead of using the Gaussian window g given by

$$g(x) = e^{-|x|^2/2}, \quad x \in \mathbb{R}^n,$$

in (1.1), we can take another suitable function φ in $L^2(\mathbb{R}^n)$ and then we have the short-time Fourier transform $G_\varphi f$ of a signal f with respect to the window φ given by

$$(G_\varphi f)(b,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \overline{\varphi(x-b)}\, dx, \quad b,\xi \in \mathbb{R}^n. \tag{1.2}$$

The formula (1.2) can be written as

$$(G_\varphi f)(b,\xi) = e^{-ib\cdot\xi} (\mathcal{F}^{-1}_{\zeta\mapsto b} f_\xi)(b), \tag{1.3}$$

where

$$f_\xi(\zeta) = \hat{f}(\zeta)\overline{\hat{\varphi}(\zeta-\xi)}, \quad \zeta \in \mathbb{R}^n.$$

The formula (1.3) tells us that we can obtain the short-time Fourier transform of a signal f by taking the Fourier transform of the signal itself, cutting it by the conjugate of the Fourier transform of the window φ and then taking the inverse Fourier transform. It is possible to invert the short-time Fourier transform using the following theorem.

Theorem 1.1 *Let f be a signal in $L^2(\mathbb{R}^n)$ and let φ be a window in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 1.$$

Then

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} (G_\varphi f)(b,\xi)\, db, \quad \xi \in \mathbb{R}^n.$$

Theorem 1.1 follows easily from the definition of the Fourier transform and the proof is exactly the same as that of Theorem 3.1 in [2].

Let us note that the Gabor transform $G_\varphi f$ of f can be written as

$$G_\varphi = (2\pi)^{-n/2}(f, M_\xi T_{-b}\varphi)_{L^2(\mathbb{R}^n)}, \quad b, \xi \in \mathbb{R}^n,$$

where $(\ ,\ )_{L^2(\mathbb{R}^n)}$ is the inner product in $L^2(\mathbb{R}^n)$, M_ξ and T_{-b} are the modulation operator and the translation operator given by

$$(M_\xi \varphi)(x) = e^{ix\cdot\xi}\varphi(x)$$

and

$$(T_{-b}\varphi)(x) = \varphi(x-b)$$

for all measurable functions φ on $\mathbb{R}^n$ and all x in $\mathbb{R}^n$. At this point, we can recall the resolution of the identity formula for the short-time Fourier transform.

Theorem 1.2 *Let φ be a window in $L^2(\mathbb{R}^n)$ such that*

$$\|\varphi\|_{L^2(\mathbb{R}^n)} = 1,$$

where $\|\ \|_{L^2(\mathbb{R}^n)}$ is the norm in $L^2(\mathbb{R}^n)$. Then for all f and g in $L^2(\mathbb{R}^n)$,

$$(f, g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (G_\varphi f)(b,\xi)\overline{(G_\varphi g)(b,\xi)}\, db\, d\xi. \tag{1.4}$$

Theorem 1.2 is the special case of Theorem 3.1 in [1] in which the matrix A is the identity matrix and can be seen as another continuous inversion formula for the short-time Fourier transform. Indeed, by (1.4), we can write

$$f = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f, M_\xi T_{-b}\varphi)_{L^2(\mathbb{R}^n)} M_\xi T_{-b}\varphi\, db\, d\xi.$$

The major drawback of the short-time Fourier transform is the fixed width of the analyzing window. Indeed, in many applications, the high frequency content of a signal is more time/space-localized than the low-frequency one. So, it is important to introduce a transform such that its analyzing window can be adapted to the frequency to be analyzed. The Stockwell transform, sometimes referred to as the S-transform in the literature, Sf of a signal $f \in L^2(\mathbb{R})$ is introduced in [3] by Stockwell, Mansinha and Lowe as

$$(Sf)(b,\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x)|\xi| e^{-(\xi(x-b))^2/2} dx, \quad b,\xi \in \mathbb{R}. \tag{1.5}$$

Since then, the Stockwell transform has been used in various applications such as geophysics, medical imaging, electrical engineering and many others.

Many extensions of the Stockwell transform have been proposed in recent years. See, for example, [4] [5] and [6]. A natural extension should be one that acts like the short-time Fourier transform generalizing the Gabor transform. Our preference is the one-dimensional modified Stockwell transforms studied in [7, 8, 9]. To wit, let f be a signal in $L^2(\mathbb{R})$ and let φ be a window in $L^2(\mathbb{R})$. Then for $1 \le s < \infty$, the modified Stockwell transform $S_{s,\varphi} f$ of the signal f with respect to the window φ is given by

$$(S_{s,\varphi})f)(b,\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)|\xi|^{1/s} \overline{\varphi(\xi(x-b))}\, dx, \quad b,\xi \in \mathbb{R}. \tag{1.6}$$

The original Stockwell transform in (1.5) is the special case when $s = 1$ and φ is the Gaussian function g. In [2], an inversion formula similar to Theorem 1.1 is stated. Precisely, it is the following theorem.

Theorem 1.3 *Let f be a signal in $L^2(\mathbb{R})$ and let φ be a window in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1.$$

Then

$$\hat{f}(\xi) = \int_{\mathbb{R}} (S_\varphi f)(b,\xi)\, db, \quad \xi \in \mathbb{R}.$$

Theorem 1.3 is still valid for the modified Stockwell transform given in (1.6). More precisely, the conclusion should now be replaced by

$$\int_{\mathbb{R}} (S_{s,\varphi} f)(b,\xi)\, db = |\xi|^{(1/s)-1} \hat{f}(\xi), \quad \xi \in \mathbb{R} \setminus \{0\}.$$

See Theorem 8 in [9].

It is obvious that

$$(S_{s,\varphi}f)(b,\xi) = (2\pi)^{-1/2}(f, M_\xi T_{-b} D_{s,\xi^{-1}}\varphi)_{L^2(\mathbb{R})},$$

where $D_{s,\xi^{-1}}$ is the one-dimensional dilation operator given by

$$(D_{s,\xi^{-1}}\varphi)(x) = |\xi|^{1/s}\varphi(\xi x) \tag{1.7}$$

for all measurable functions φ on $\mathbb{R}$ and all x in $\mathbb{R}$. We can think of the Stockwell transform as a short-time Fourier transform in which the width of the analyzing window varies in accordance with the frequency to be analyzed. The following resolution of the identity formula can be found in [9, 10].

Theorem 1.4 *Let φ be a window in $L^2(\mathbb{R})$ such that*

$$c_\varphi = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi+1|} < \infty.$$

Then for all f and g in $L^2(\mathbb{R})$,

$$c_\varphi (f,g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} (S_{s,\varphi}f)(b,\xi)\overline{(S_{s,\varphi}g)(b,\xi)}db\, \frac{d\xi}{|\xi|^{(2/s)-1}}.$$

Theorem 1.4 points out the similarity of the modified Stockwell transform with the wavelet transform that we now recall. The wavelet transform $\Omega_\varphi f$ of a signal f in $L^2(\mathbb{R})$ with respect to the window φ in $L^2(\mathbb{R})$ is given by

$$(\Omega_\varphi f)(b,a) = \int_{\mathbb{R}} f(x)|a|^{-1/2}\overline{\varphi(a^{-1}(x-b))}\, dx, \quad b\in\mathbb{R}, a\in\mathbb{R}\setminus\{0\}.$$

The modified Stockwell transform and the wavelet transform are related by the formula to the effect that

$$(S_{s,\varphi}f)(b,\xi) = (2\pi)^{-1/2}e^{-ib\xi}|\xi|^{(1/s)-(1/2)}(\Omega_\psi f)(b,1/\xi), \quad b\in\mathbb{R}, \xi\in\mathbb{R}\setminus\{0\},$$

where

$$\psi(x) = e^{ix}\varphi(x), \quad x\in\mathbb{R}.$$

The analog of Theorem 1.4 for the wavelet transform is the following theorem, which is a well-known result, e.g., Proposition 2.4.1 in [11].

Theorem 1.5 *Let φ be a window in $L^2(\mathbb{R})$ such that*

$$c_\varphi = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\xi|^2} < \infty.$$

Then for all f and g in $L^2(\mathbb{R})$,

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} (\Omega_\varphi f)(b, a) \overline{(\Omega_\varphi g)(b, a)} db \, \frac{da}{a^2}.$$

The aim of this paper is to obtain multi-dimensional analogs of the modified Stockwell transform in (1.6) and give continuous inversion formulas for them.

In Section 2, we introduce multi-dimensional modified Stockwell transforms, which modify the one given in [1]. In Section 3, a result on the continuous inversion formula in [1] is revisited and shown to be only valid for dimensions equal to 1, 2, 4 and 8 in view of a topological result in [13]. A new continuous inversion formula for this multi-dimensional Stockwell transform is stated and proved in Section 4. This formula includes Theorem 1.2, Theorem 1.4 as special cases.

It is also worth pointing out the relations of the results in this paper with those in [1]. Theorem 3.1 in this paper is Theorem 2.3 in [1] and examples of 2×2, 4×4 and 8×8 matrices satisfying the hypotheses of the theorem are given in [1]. We explain in this paper why the hypotheses cannot be satisfied for dimensions other than 1, 2, 4 and 8. The new result in this paper is Theorem 4.3, which has no obvious connection with Theorem 3.1 in this paper. The multi-dimensional Gabor transforms in Section 3 of [1] and the multi-dimensional non-isotropic Stockwell transforms in [1] are examples to which Theorem 4.3 can be applied.

One-dimensional modified Stockwell transforms in the context of time-frequency analysis have been studied in [7, 8, 9]. In an era in which mathematical sciences develop unprecedentally fast, it is envisaged that multi-dimensional modified Stockwell transforms are additional useful tools to understand and analyze the architecture of large data sets gathered from investigations in many areas.

2 Multi-Dimensional Modified Stockwell Transforms

The starting point is to extend the one-dimensional dilation operator given by (1.7). Let $A \in \mathrm{GL}(n, \mathbb{R})$. Then for $1 \leq s < \infty$, the multi-dimensional dilation operator $D_{s,A}$ is defined by

$$(D_{s,A}\varphi)(x) = |\det A|^{-1/s}\varphi(A^{-1}x), \quad x \in \mathbb{R}^n,$$

for all measurable functions φ on $\mathbb{R}^n$. If $s = 2$, then $D_{2,A}$ is a unitary operator on $L^2(\mathbb{R}^n)$.

Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be given by

$$\mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$$

and let $\varphi \in L^2(\mathbb{R}^n)$. Then for $1 \leq s < \infty$, we define the multi-dimensional modified Stockwell transform $S_{s,A,\varphi}f$ of a signal f in $L^2(\mathbb{R}^n)$ by

$$(S_{s,A,\varphi}f)(b, \xi) = (2\pi)^{-n/2}|\det A_\xi|^{-1/s} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\overline{\varphi(A_\xi^{-1}(x - b))}\, dx \quad (2.1)$$

for all $(b, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. We note that for all b and ξ in $\mathbb{R}^n$,

$$\begin{aligned} &(S_{s,A,\varphi}f)(b, \xi) \\ = \; &|\det A_\xi|^{(1/2)-(1/s)}(S_{2,A,\varphi}f)(b, \xi) \\ = \; &(2\pi)^{-n/2}|\det A_\xi|^{(1/2)-(1/s)}(f, M_\xi T_{-b} D_{2,A_\xi}\varphi)_{L^2(\mathbb{R}^n)}. \end{aligned} \quad (2.2)$$

The modified Stockwell transform $S_{s,A,\varphi}f$ of a signal $f \in L^2(\mathbb{R}^n)$ can be expressed in terms of the Fourier transform $\hat{f}$ of the signal f.

Proposition 2.1 *Let $f, \varphi \in L^2(\mathbb{R}^n)$. Then for $1 \leq s < \infty$,*

$$(S_{s,A,\varphi}f)(b, \xi) = |\det A_\xi|^{1-(1/s)}e^{-ib\cdot\xi}(\mathcal{F}^{-1}_{\zeta\mapsto b}f_{\xi,A_\xi})(b), \quad b, \xi \in \mathbb{R}^n,$$

where

$$f_{\xi,A_\xi}(\zeta) = \hat{f}(\zeta)\overline{\hat{\varphi}(A_\xi^t(\zeta - \xi))}, \quad \zeta \in \mathbb{R}^n.$$

Proof For all b and $\xi \in \mathbb{R}^n$,

$$\mathcal{F}T_{-b}M_\xi D_{2,A_\xi}\varphi = M_{-b}T_{-\xi}D_{2,(A_\xi^{-1})^t}\mathcal{F}\varphi.$$

So, by Plancherel's formula,

$$\begin{aligned}
& (f, T_{-b}M_\xi D_{2,A_\xi}\varphi)_{L^2(\mathbb{R}^n)} \\
= \; & (\mathcal{F}f, \mathcal{F}T_{-b}M_\xi D_{2,A_\xi}\varphi)_{L^2(\mathbb{R}^n)} \\
= \; & (\mathcal{F}f, M_{-b}T_{-\xi}D_{2,(A_\xi^{-1})^t}\mathcal{F}\varphi)_{L^2(\mathbb{R}^n)} \\
= \; & |\det A_\xi|^{1/2}\int_{\mathbb{R}^n}\hat{f}(\zeta)e^{ib\cdot\zeta}\overline{\hat{\varphi}(A_\xi^t(\zeta-\xi))}\,d\zeta \\
= \; & (2\pi)^{n/2}|\det A_\xi|^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n}\hat{f}(\zeta)e^{ib\cdot\zeta}\overline{\hat{\varphi}(A_\xi^t(\zeta-\xi))}\,d\zeta \\
= \; & (2\pi)^{n/2}|\det A_\xi|^{1/2}(\mathcal{F}^{-1}_{\zeta\mapsto b}f_{\xi,A_\xi})(b), \quad b \in \mathbb{R}^n.
\end{aligned}$$

Since

$$(S_{2,A,\varphi}f)(b,\xi) = (2\pi)^{-n/2}e^{-ib\cdot\xi}(f, T_{-b}M_\xi D_{2,A_\xi}\varphi)_{L^2(\mathbb{R}^n)}, \quad b,\xi \in \mathbb{R}^n,$$

it follows from (2.2) that

$$\begin{aligned}
& (S_{s,A,\varphi}f)(b,\xi) \\
= \; & |\det A_\xi|^{(1/2)-(1/s)}(S_{2,A,\varphi}f)(b,\xi) \\
= \; & |\det A_\xi|^{(1/2)-(1/s)}(2\pi)^{-n/2}e^{-ib\cdot\xi}(f, T_{-b}M_\xi D_{2,A_\xi}\varphi)_{L^2(\mathbb{R}^n)} \\
= \; & |\det A_\xi|^{(1/2)-(1/s)}(2\pi)^{-n/2}e^{-ib\cdot\xi}(2\pi)^{n/2}|\det A_\xi|^{1/2}(\mathcal{F}^{-1}_{\zeta\mapsto b}f_{\xi,A_\xi})(b) \\
= \; & |\det A_\xi|^{1-(1/s)}e^{-ib\cdot\xi}(\mathcal{F}^{-1}_{\zeta\mapsto b}f_{\xi,A_\xi})(b), \quad b,\xi \in \mathbb{R}^n.
\end{aligned}$$

□

Remark 2.2 Proposition 2.1 tells us that we can obtain the modified Stockwell transform of a signal by taking the Fourier transform of the signal, cutting it by means of the conjugate of a dilated Fourier transform of a window and then taking the inverse Fourier transform. This suggests the possibility of developing a fast algorithm to compute the modified Stockwell transform as the one developed in [12].

3 The Case for s=1

A recall of Theorem 2.3 in [1] is in order. The multi-dimensional Stockwell transform therein is the modified Stockwell transform with $s = 1$. We call it Theorem 3.1 in this paper.

Theorem 3.1 *Suppose that the mapping* $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ *given by*

$$\mathbb{R}^n \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$$

satisfies the following two conditions.
(a) *If* $\eta = A_\xi^t \zeta$ *for some* $\zeta \in \mathbb{R}^n$, *then there exist two positive functions* f_1 *and* f_2 *on* $\mathbb{R}^n$ *such that*

$$\left| \det \left(\frac{\partial \eta}{\partial \xi} \right) \right| = \frac{f_1(\eta)}{f_2(\xi)}, \quad \xi, \eta \in \mathbb{R}^n,$$

where $\frac{\partial \eta}{\partial \xi}$ *is the Jacobian matrix of* η *with respect to* ξ.
(b) *There exists a vector* $v \in \mathbb{R}^n$ *such that*

$$A_\xi^t \xi = v.$$

Let $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} |\hat{\varphi}(\eta)|^2 \frac{d\eta}{f_1(\eta + v)} < \infty.$$

Then

$$c_\varphi (f, g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (S_{1,A,\varphi} f)(b, \xi) \overline{(S_{1,A,\varphi} g)(b, \xi)} \, db \frac{d\xi}{f_2(\xi)}$$

for all f *and* g *in* $L^2(\mathbb{R}^n)$.

Remark 3.2 The following matrix functions $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ for $n = 2, 4, 8$ satisfy Theorem 3.1 if each is divided by $|\xi|^2$.

$$A_\xi = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}$$

$$A_\xi = \begin{pmatrix} \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 \\ \xi_2 & \xi_1 & -\xi_4 & \xi_3 \\ \xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ \xi_4 & -\xi_3 & \xi_2 & \xi_1 \end{pmatrix}$$

$$A_\xi = \begin{pmatrix} \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6 & -\xi_7 & -\xi_8 \\ \xi_2 & \xi_1 & -\xi_4 & \xi_3 & -\xi_6 & \xi_5 & \xi_8 & -\xi_7 \\ \xi_3 & \xi_4 & \xi_1 & -\xi_2 & -\xi_7 & -\xi_8 & \xi_5 & \xi_6 \\ \xi_4 & -\xi_3 & \xi_2 & \xi_1 & -\xi_8 & \xi_7 & -\xi_6 & \xi_5 \\ \xi_5 & \xi_6 & \xi_7 & \xi_8 & \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 \\ \xi_6 & -\xi_5 & \xi_8 & -\xi_7 & \xi_2 & \xi_1 & \xi_4 & -\xi_3 \\ \xi_7 & -\xi_8 & -\xi_5 & \xi_6 & \xi_3 & -\xi_4 & \xi_1 & \xi_2 \\ \xi_8 & \xi_7 & -\xi_6 & -\xi_5 & \xi_4 & \xi_3 & -\xi_2 & \xi_1 \end{pmatrix}$$

That it is not possible to extend Theorem 3.1 to dimensions other than 1, 2, 4 and 8 is due to the following proposition.

Proposition 3.3 *Let $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$. Then there does not exist a continuous mapping*

$$\mathbb{S}^{n-1} \ni \xi \mapsto A_\xi \in \mathrm{GL}(n, \mathbb{R})$$

such that $A_\xi \xi$ is parallel to to the vector $e_1 \in \mathbb{R}^n$, where e_1 has 1 in the first entry and zeros elsewhere.

The proof of Proposition 3.3 can be found in [14] and it is based on the paper [13] by Bott and Milnor. In view of Proposition 3.3, part (b) of the hypotheses of Theorem 3.1 cannot be satisfied for $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$.

4 Continuous Inversion Formulas

New continuous inversion formulas for multi-dimensional modified Stockwell transforms can now be given. We begin with a formula on the Fourier transform.

Proposition 4.1 *Let f be a signal in $L^2(\mathbb{R}^n)$ and let $\varphi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 1.$$

Then

$$\hat{f}(\xi) = |\det A_\xi|^{(1/s)-1} \int_{\mathbb{R}^n} (S_{s,A,\varphi} f)(b, \xi)\, db, \quad \xi \in \mathbb{R}^n.$$

Proof Using the Fubini theorem and the change of variable from x to y via

$$y = A_\xi^{-1}(x - b),$$

we get

$$\begin{aligned}
& \int_{\mathbb{R}^n} (S_{s,A,\varphi} f)(b, \xi)\, db \\
= & \int_{\mathbb{R}^n} \left((2\pi)^{-n/2} |\det A_\xi|^{-1/s} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \overline{\varphi(A_\xi^{-1}(x-b))} dx \right) db \\
= & (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \overline{\varphi(A_\xi^{-1}(x-b))} |\det A_\xi|^{-1/s} dx\, db \\
= & (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) |\det A_\xi|^{-1/s} \left(\int_{\mathbb{R}^n} \overline{\varphi(A_\xi^{-1}(x-b))} db \right) dx \\
= & (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) |\det A_\xi|^{1-(1/s)} \left(\int_{\mathbb{R}^n} \overline{\varphi(y)}\, dy \right) dx \\
= & |\det A_\xi|^{1-(1/s)} \hat{f}(\xi), \quad \xi \in \mathbb{R}^n.
\end{aligned}$$

□

In order to have new resolution of the identity formulas for a wider class of modified Stockwell transforms, we first give the following lemma from [15].

Lemma 4.2 *Let $A : \mathbb{R}^n \to \mathrm{GL}(n, \mathbb{R})$ be a piecewise differentiable function such that we can find a fixed* (1,2)*-tensor F and a fixed* (1,1)*-tensor G for which*

$$(A_\xi^t)^{-1} = (F^i_{jl} \xi^l + G^i_j)_{1 \le i,j \le n},$$

i.e.,

$$(A_\xi^t)^{-1} = \begin{pmatrix} \sum_{l=1}^n F_{1l}^1 \xi^l & \cdots & \sum_{l=1}^n F_{nl}^1 \xi^l \\ \vdots & \vdots & \vdots \\ \sum_{l=1}^n F_{1l}^n \xi^l & \cdots & \sum_{l=1}^n F_{nl}^n \xi^l \end{pmatrix} + \begin{pmatrix} G_1^1 & \cdots & G_n^1 \\ \vdots & \vdots & \vdots \\ G_1^n & \cdots & G_n^n \end{pmatrix}.$$

Then

$$\int_{\mathbb{R}^n} |\hat{\varphi}(A_\xi^t(\zeta - \xi))|^2 |\det A_\xi| \, d\xi = \int_{\eta_\zeta(\mathbb{R}^n)} |\hat{\varphi}(\eta)|^2 \frac{d\eta}{|\det (F_{jk}^i \eta^j + \delta_k^i)|},$$

where $\eta_\zeta : \mathbb{R}^n \to \mathbb{R}^n$ *is given by*

$$\eta_\zeta(\xi) = A_\xi^t(\zeta - \xi), \quad \xi \in \mathbb{R}^n.$$

A poof of Lemma 4.2 can be found in [15]. For the sake of being self-contained, we give a more expanded proof than that in [15] filling in the details.

Proof of Lemma 4.2 We begin with the $(1,1)$-tensor

$$Q_i^j = (A_\xi^t)_i^j$$

and its inverse

$$S_j^i = ((A_\xi^t)^{-1})_j^i.$$

Then for $\xi \in \mathbb{R}^n$, we define the function $\eta : \mathbb{R}^n \times \mathbb{R}^n$ by

$$\mathbb{R}^n \ni \zeta \mapsto \eta_\zeta(\xi),$$

where

$$(\eta_\zeta(\xi))^j = Q_i^j (\zeta - \xi)^i.$$

If we denote the Jacobian $\frac{\partial \eta}{\partial \xi}$ by $J_\eta(\xi)$, then

$$\begin{aligned} (J_\eta(\xi))_k^j &= \partial_k Q_i^j (\zeta - \xi)^i + Q_i^j \partial_k (\zeta - \xi)^i \\ &= \partial_k Q_i^j (\zeta - \xi)^i - Q_i^j \delta_k^i \\ &= Q_i^j (S_j^i \partial_k Q_i^j (\zeta - \xi)^i - \delta_k^i) \\ &= Q_i^j (-(\partial_k S_j^i) Q_i^j (\zeta - \xi)^i - \delta_k^i) \\ &= Q_i^j (-(\partial_k S_j^i) \eta^j - \delta_k^i). \end{aligned}$$

Thus,

$$S^i_j = ((A^{-1}_\xi)^t)^i_j = F^i_{jl}\xi^l + G^i_j.$$

So,

$$\partial_k S^i_j = F^i_{jl}\delta^l_k = F^i_{jk}$$

and

$$(J_\eta(\xi))^j_k = Q^j_i(-F^i_{jk}\eta^j - \delta^i_k).$$

Observing that

$$\det Q^j_i = \det A^t_\xi = \det A_\xi$$

and

$$\det (J_\eta(\xi))^j_k = (-1)^n(\det A_\xi)\det (F^i_{jk}\eta^j + \delta^i_k),$$

we get

$$d\eta = |\det (J_\eta(\xi))^j_k|\, d\xi = |\det A_\xi|\, |\det (F^i_{jk}\eta^j + \delta^i_k)|\, d\xi.$$

Hence

$$\int_{\mathbb{R}^n} |\hat{\varphi}(A^t_\xi(\zeta - \xi))|^2 |\det A_\xi|\, d\xi = \int_{\eta_\zeta(\mathbb{R}^n)} |\hat{\varphi}(\eta)|^2 \frac{d\eta}{|\det (F^i_{jk}\eta^j + \delta^i_k)|}.$$

This completes the proof of Lemma 4.2. □

We can now give a new resolution of the identity formula for modified Stockwell transforms.

Theorem 4.3 *Let the hypotheses of Lemma* 4.2 *be satisfied. Moreover, suppose that*

$$\eta_\zeta(\mathbb{R}^n) = \mathbb{R}^n$$

for all $\zeta \in \mathbb{R}^n$. *Let* $\varphi \in L^2(\mathbb{R}^n)$ *be such that*

$$c_\varphi = \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 \frac{d\xi}{|\det (F^i_{jk}\xi^j + \delta^i_k)|} < \infty.$$

Then

$$c_\varphi (f, g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (S_{s,A,\varphi} f)(b, \xi)\overline{(S_{s,A,\varphi} g)(b, \xi)}\, db\, |\det A_\xi|^{(2/s)-1} d\xi$$

for all f *and* g *in* $L^2(\mathbb{R}^n)$.

Proof Using Proposition 2.1, the Fubini theorem, Plancherel's formula and Lemma 4.2, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(S_{s,A,\varphi}f)(b,\xi)\overline{(S_{s,A,\varphi}g)(b,\xi)}\,db\,|\det A_\xi|^{(2/s)-1}d\xi\\
=\;&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\det A_\xi|^{1-(1/s)}e^{-ib\cdot\xi}(F^{-1}_{\zeta\mapsto b}f_{\xi,A_\xi})(b)\\
&\times\overline{|\det A_\xi|^{1-(1/s)}e^{-ib\cdot\xi}(\mathcal{F}^{-1}_{\zeta\mapsto b}g_{\xi,A_\xi})(b)}\,db\,|\det A_\xi|^{(2/s)-1}\,d\xi\\
=\;&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathcal{F}^{-1}_{\zeta\mapsto b}f_{\xi,A_\xi})(b)\overline{(\mathcal{F}^{-1}_{\zeta\mapsto b}g_{\xi,A_\xi})(b)}\,db\,|\det A_\xi|\,d\xi\\
=\;&\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}f_{\xi,A_\xi}(\zeta)\overline{g_{\xi,A_\xi}(\zeta)}\,d\zeta\right)|\det A_\xi|\,d\xi\\
=\;&\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\hat{f}(\zeta)\overline{\hat{\varphi}(A_\xi^t(\zeta-\xi))}\overline{\hat{g}(\zeta)\overline{\hat{\varphi}(A_\xi^t(\zeta-\xi))}}\,d\zeta\right)|\det A_\xi|\,d\xi\\
=\;&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\zeta)\overline{\hat{g}(\zeta)}\left(\int_{\mathbb{R}^n}|\hat{\varphi}(A_\xi^t(\zeta-\xi))|^2|\det A_\xi|\,d\xi\right)d\zeta\\
=\;&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\zeta)\overline{\hat{g}(\zeta)}\left(\int_{\eta_\zeta(\mathbb{R}^n)}|\hat{\varphi}(\eta)|^2\frac{d\eta}{|\det(F^i_{jk}\eta^j+\delta^i_k)|}\right)d\zeta\\
=\;&c_\varphi\int_{\mathbb{R}^n}\hat{f}(\zeta)\overline{\hat{g}(\zeta)}\,d\zeta\\
=\;&c_\varphi(f,g)_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

□

Remark 4.4 Theorem 4.3 in this paper contains Theorem 3.1 in [1] (constant matrices) and Corollary 4.1 in [1] (diagonal matrices) as special cases. Theorem 4.3 is certainly applicable to the constant and diagonal matrices for all dimensions n. It does not, however, include Theorem 3.1 in this paper as a special case. The matrix-valued functions given by the 2×2, 4×4 and 8×8 matrices in Remark 3.2 satisfy the hypotheses of Theorem 4.3. It should be pointed out that Lemma 4.2 and Theorem 4.3 are in [16].

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