PSEUDO-DIFFERENTIAL OPERATORS FOR WEYL TRANSFORMS

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Pseudo-differential operators with operator-valued symbols for Weyl transforms are introduced. We give suitable conditions on the symbols for which these operators are in the trace class and give a trace formula for them.

Keywords: Weyl transforms, pseudo-differential operators, symbols, noncommutative quantizations, L^2-boundedness, Hilbert–Schmidt operators, trace class operators, traces

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1. Weyl Transforms

In this paper we identify $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with the complex space $\mathbb{C}^n$ via the obvious identification

$$\mathbb{R}^{2n} \ni (q,p) \leftrightarrow q + ip \in \mathbb{C}^n.$$

Let $q = (q_1, q_2, \dots, q_n)$ and $p = (p_1, p_2, \dots, p_n)$ be points in $\mathbb{R}^n$, and let f be a measurable function on $\mathbb{R}^n$. Then we define the function $\rho(q,p)f$ on $\mathbb{R}^n$ by

$$(\rho(q,p)f)(x) = e^{iq\cdot x + i(q\cdot p)/2} f(x+p), \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

where

$$q \cdot x = \sum_{j=1}^n q_j x_j$$

and

$$q \cdot p = \sum_{j=1}^n q_j p_j.$$

It is clear that $\rho(q,p) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator for all q and p in $\mathbb{R}^n$.

Let f and g be in $L^2(\mathbb{R}^n)$. Then we define the function $V(f,g)$ on $\mathbb{R}^{2n}$ by

$$V(f,g)(q,p) = (2\pi)^{-n/2} (\rho(q,p)f, g)_{L^2(\mathbb{R}^n)}.$$

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We call $V(f,g)$ the Fourier–Wigner transform of f and g and the Wigner transform $W(f,g)$ of f and g is defined by

$$W(f,g) = V(f,g)^\wedge,$$

where $V(f,g)^\wedge$ is the Fourier transform of $V(f,g)$. It should be noted that the Fourier transform $\hat{F}$ of a function F in $L^1(\mathbb{R}^N)$ is taken to be

$$\hat{F}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} F(x)\, dx, \quad \xi \in \mathbb{R}^N.$$

An integral representation of $W(f,g)$ is given by the following theorem.

Theorem 1.1. *Let f and g be in $L^2(\mathbb{R}^n)$. Then*

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp$$

for all x and ξ in $\mathbb{R}^n$.

Let $u \in L^1(\mathbb{C}^n)$. Then we define the Weyl transform of u to be the bounded linear operator $W_u : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$(W_u f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x,\xi) W(f,g)(x,\xi)\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$. That $W_u : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator is easy to see. Indeed, for all φ and ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, by Theorem 1.1,

$$|W(\varphi,\psi)(x,\xi)| \le \left(\frac{2}{\pi}\right)^{n/2} \|\varphi\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)}, \quad x, \xi \in \mathbb{R}^n.$$

Thus,

$$\begin{aligned} |(W_u \varphi, \psi)_{L^2(\mathbb{R}^n)}| &\le (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x,\xi)|\, |W(\varphi,\psi)(x,\xi)| dx\, d\xi \\ &\le \pi^{-n} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x,\xi)|\, dx\, d\xi \right) \|\varphi\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)} \\ &= \pi^{-n} \|u\|_{L^1(\mathbb{C}^n)} \|\varphi\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R})}. \end{aligned}$$

At this point, it is good to have a pause and reflect that Weyl transforms W_u have hitherto been studied as symmetric forms of pseudo-differential operators with symbols u. See [7, 8] in this connection. This may be considered as the first generation in which the symbols of pseudo-differential operators are functions.

A more in-depth study of Weyl transforms requires a recall of Hilbert–Schmidt operators and trace class operators [5, 9]. Let X be an infinite-dimensional complex and separable Hilbert space. Let A be a compact operator on X. If we denote the adjoint of A by A^*, then $\sqrt{A^*A}$ is a compact and self-adjoint operator on X. By the spectral theorem, we can find an orthonormal basis $\{\varphi_k : k = 1, 2, \dots\}$ for X consisting of eigenvectors of $\sqrt{A^*A}$. For $k = 1, 2, \dots,$ let s_k be the eigenvalue of

$\sqrt{A^*A}$ corresponding to the eigenvector φ_k. Then for $1 \le p < \infty$, A is said to be in the Schatten–von Neumann class S_p if

$$\sum_{k=1}^{\infty} s_k^p < \infty.$$

If $A \in S_p$, then the norm $\|A\|_{S_p}$ of A is defined by

$$\|A\|_{S_p} = \left\{ \sum_{k=1}^{\infty} s_k^p \right\}^{1/p}.$$

The class S_∞ is simply the C^*-algebra of all bounded linear operators on X. It is obvious that

$$S_1 \subset S_2 \subset \cdots \subset S_\infty$$

and

$$\|\ \|_{S_1} \ge \|\ \|_{S_2} \ge \cdots \ge \|\ \|_{S^\infty}.$$

Of particular importance are the operators in S_1 and S_2 known as trace class operators and Hilbert–Schmidt operators respectively. The following facts on S_1 and S_2 are used in this paper.

Theorem 1.2. *For $1 \le p < \infty$, S_p is a two-sided ideal in S_∞.*

Theorem 1.3. *A compact operator A on X is in S_1 if and only if there exist two operators B and C in S_2 such that*

$$A = BC.$$

Theorem 1.4. *Let $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a compact operator. Then $A \in S_2$ if and only if there exists a function k in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$(Af)(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\, dy, \quad x \in \mathbb{R}^n.$$

Moreover, if $A \in S_2$, then

$$\|A\|_{S_2} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |k(x,y)|^2 dx\, dy \right)^{1/2}.$$

Let $k \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we denote the corresponding Hilbert–Schmidt operator on $L^2(\mathbb{R}^n)$ by A_k.

Theorem 1.5. *Let g and h be functions in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$A_g A_h = A_{g \circ h},$$

where

$$(g \circ h)(z,w) = \int_{\mathbb{R}^n} g(z,\zeta) h(\zeta,w)\, d\zeta, \quad z,w \in \mathbb{R}^n.$$

A fundamental result in the analysis of Weyl transforms is the following theorem.

Theorem 1.6. *Let $u \in L^2(\mathbb{C}^n)$. Then $W_u : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert–Schmidt operator and*

$$\|W_u\|_* \le \|W_u\|_{S_2} = (2\pi)^{-n/2}\|u\|_{L^2(\mathbb{C}^n)},$$

where $\| \, \|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$.

The converse of Theorem 1.6 is also true. In other words, every Hilbert–Schmidt operator on $L^2(\mathbb{R}^n)$ is a Weyl transform W_σ with symbol σ in $L^2(\mathbb{C}^n)$.

Another important result is the following integral representation of the Weyl transform.

Theorem 1.7. *Let $u \in L^2(\mathbb{C}^n)$. Then for all f in $L^2(\mathbb{R}^n)$,*

$$W_u f = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{u}(q,p)\rho(q,p) f \, dq \, dp.$$

The starting point of this paper is the following inversion formula for the Weyl transform [1, 6].

Theorem 1.8. *Let u be a Schwartz function on $\mathbb{C}^n$. Then*

$$u(z) = \operatorname{tr}(\rho(z)^* W_{\check{u}}), \quad z \in \mathbb{C}^n,$$

where $\check{u}$ is the inverse Fourier transform of u.

Let $\sigma : \mathbb{C}^n \to B(L^2(\mathbb{R}^n))$. Then we define the pseudo-differential operator $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ by

$$(T_\sigma u)(z) = \operatorname{tr}(\rho(z)^* \sigma(z) W_{\check{u}}), \quad z \in \mathbb{C}^n,$$

for all Schwartz functions u on $\mathbb{C}^n$. We call T_σ the pseudo-differential operator for the Weyl transform with operator-valued symbol or just simply symbol σ. As pseudo-differential operators with symbols given by operators instead of functions, we may consider these operators to be in the second generation of pseudo-differential operators. The usefulness of these second-generation pseudo-differential operators lies in the realm of noncommutative quantizations, which we briefly sketch here. See [2] for details. To wit, in classical (commutative) quantizations, observables given by functions of positions x and momentum ξ in $\mathbb{R}^n$ are quantized to give rise to linear operators on Hilbert spaces in general and pseudo-differential operators on $L^2(\mathbb{R}^n)$ in particular. The viewpoint to be popularized here is that the classical observables are first quantized to Weyl transforms, which are noncommutative linear operators on $L^2(\mathbb{R}^n)$, and then pseudo-differential operators on $L^2(\mathbb{R}^n)$ are built as quantizations of the first-generation quantizations.

The aim of this paper is to give conditions for which these operators are Hilbert–Schimdt and in the trace class, and give a trace formula for these trace class operators.

In Section 2 we give the boundedness, and in Section 3 the Hilbert–Schmidt property of pseudo-differential operators for the Weyl transform on $L^2(\mathbb{C}^n)$. In Section 4, we give a class of trace class pseudo-differential operators for the Weyl transform on $L^2(\mathbb{C}^n)$ and give a trace formula for these operators.

Related results can be found in the paper [3].

2. L^2-Boundedness

Theorem 2.1. *Let $\sigma : \mathbb{C}^n \to S_2$ be such that*

$$\int_{\mathbb{C}^n} \|\sigma(z)\|_{S_2}^2 dz < \infty.$$

Then the pseudo-differential operator $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a bounded linear operator. Moreover,

$$\|T_\sigma\|_{**} \le (2\pi)^{-n/2} \left(\int_{\mathbb{C}^n} \|\sigma(z)\|_{S_2}^2 dz \right)^{1/2},$$

where $\| \, \|_{**}$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{C}^n)$.

Proof For all functions u in $L^2(\mathbb{C}^n)$,

$$\begin{aligned} \|T_\sigma u\|_{L^2(\mathbb{C}^n)}^2 &= \int_{\mathbb{C}^n} |\mathrm{tr}(\rho(z)^* \sigma(z) W_{\check{u}})|^2 dz \\ &\le \int_{\mathbb{C}} \|\sigma(z)\|_{S_2}^2 \|W_{\check{u}}\|_{S_2}^2 dz \\ &= (2\pi)^{-n} \left(\int_{\mathbb{C}^n} \|\sigma(z)\|_{S_2}^2 dz \right) \|u\|_{L^2(\mathbb{C}^n)}^2. \end{aligned}$$

□

Remark 2.1. That some conditions on the mapping $\sigma : \mathbb{C}^n \to S_2$ like the one in the hypothesis of Theorem 2.1 is required can be seen from the following example.

Example 2.1. Let $\alpha \in L^2(\mathbb{C}^n)$ and let $\sigma : \mathbb{C}^n \to S_2$ be the symbol given by

$$\sigma(z) = \rho(z) W_\alpha, \quad z \in \mathbb{C}^n.$$

Then for all functions u in $L^2(\mathbb{C}^n)$,

$$(T_\sigma u)(z) = \mathrm{tr}(W_{\check{u}}), \quad z \in \mathbb{C}^n,$$

and obviously, $T_\sigma u \notin L^2(\mathbb{C}^n)$ unless $\mathrm{tr}(W_{\check{u}}) = 0$.

We need the following proposition for the rest of the paper. It states that different symbols give different pseudo-differential operators.

Theorem 2.2. *Let $\sigma : \mathbb{C}^n \to S_2$ be such that*

$$\int_{\mathbb{C}^n} \|\sigma(z)\|_{S_2}^2 dz < \infty.$$

Furthermore, suppose that the mapping

$$\mathbb{C}^n \ni z \mapsto \rho(z)^* \sigma(z) \in S_2 \tag{2.1}$$

is weakly continuous. Then

$$T_\sigma = 0 \Rightarrow \sigma = 0.$$

Proof For all u in $L^2(\mathbb{C}^n)$, we get

$$(T_\sigma u)(z) = \operatorname{tr}(\rho(z)^*\sigma(z)W_{\check{u}}) = 0, \quad z \in \mathbb{C}^n.$$

Now, for all z in $\mathbb{C}^n$, let u_z be the function in $L^2(\mathbb{C}^n)$ such that

$$W_{(u_z)^\vee} = \rho(z)\sigma(z)^*.$$

Then for all w in $\mathbb{C}^n$,

$$(T_\sigma u_z)(w) = \operatorname{tr}(\rho(w)^*\sigma(w)\rho(z)\sigma(z)^*). \tag{2.2}$$

Now, $T_\sigma u_z$ is continuous on $\mathbb{C}^n$. Indeed, let $z_0 \in \mathbb{C}^n$. Then by the weak continuity of the mapping (2.1),

$$\operatorname{tr}(\rho(w)^*\sigma(w)\rho(z)\sigma(z)^*) \to \operatorname{tr}(\rho(z_0)^*\sigma(z_0)\rho(z)\sigma(z)^*)$$

as $w \to z_0$ in $\mathbb{C}^n$. If we let $w = z$ in (2.2), then

$$(T_\sigma u_z)(z) = \operatorname{tr}(\sigma(z)\sigma(z)^*) = \|\sigma(z)\|_{S_2}^2 = 0.$$

Thus, $\sigma(z) = 0$ for all z in $\mathbb{C}^n$, as asserted. □

3. Hilbert–Schmidt Operators

Theorem 3.1. *Let $\sigma : \mathbb{C}^n \to S_2$ be such that the hypotheses of Theorem 2.2 are satisfied. Then the pseudo-differential operator $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a Hilbert–Schmidt operator if and only if*

$$\sigma(z) = \rho(z)W_{\alpha(z)}, \quad z \in \mathbb{C}^n,$$

where $\alpha : \mathbb{C}^n \to L^2(\mathbb{C}^n)$ is a weakly continuous mapping such that

$$\int_{\mathbb{C}^n} \|\alpha(z)\|_{L^2(\mathbb{C}^n)}^2 dz < \infty. \tag{3.1}$$

Moreover

$$\|T_\sigma\|_{S_2} = (2\pi)^{-n/2}\left(\int_{\mathbb{C}^n} \|\alpha(z)\|_{L^2(\mathbb{C}^n)}^2 dz\right)^{1/2}.$$

Proof For all functions u in $L^2(\mathbb{C}^n)$, we get by formula (3.7) in [4]

$$\begin{aligned}(T_\sigma u)(z) &= \operatorname{tr}(\rho^*(z)\sigma(z)W_{\check{u}}) \\ &= \operatorname{tr}(W_{\alpha(z)}W_{\check{u}}) \\ &= (2\pi)^{-n}\int_{\mathbb{C}^n} (\alpha(z))(w)\check{u}(w)\,dw \\ &= \int_{\mathbb{C}^n} K(z,w)\check{u}(w)\,dw,\end{aligned}$$

where

$$K(z,w) = (2\pi)^{-n}(\alpha(z))(w), \quad z, w \in \mathbb{C}^n. \tag{3.2}$$

But, using Fubini's theorem and (3.1),

$$\begin{aligned}
&\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |K(z,w)|^2 dz\, dw \\
= \;& (2\pi)^{-2n} \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |(\alpha(z))(w)|^2 dz\, dw \\
= \;& (2\pi)^{-2n} \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |(\alpha(z))(w)|^2 dw\right) dz \\
= \;& (2\pi)^{-2n} \int_{\mathbb{C}^n} \|\alpha(z)\|^2_{L^2(\mathbb{C}^n)} dz < \infty. \qquad (3.3)
\end{aligned}$$

Therefore $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a Hilbert–Schmidt operator. The converse can be proved by reversing the argument and using Theorem 2.2 Moreover, by (6.17) and Theorem 7.5 in [7], and (3.2) we get

$$\|T_\sigma\|^2_{S_2} = \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |K(z,w)|^2 dz\, dw,$$

and hence by (3.3),

$$\|T_\sigma\|_{S_2} = (2\pi)^{-n} \left\{\int_{\mathbb{C}^n} \|\alpha(z)\|^2_{L^2(\mathbb{C}^n)} dz\right\}^{1/2}.$$

□

We give in the following theorem a stronger result than Theorem 3.1.

Theorem 3.2. *Let $\sigma : \mathbb{C}^n \to B(L^2(\mathbb{R}^n))$. Then the pseudo-differential operator $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a Hilbert–Schmidt operator if and only if σ satisfies the hypotheses of Theorem* 2.2. *Moreover, if the hypotheses of Theorem* 2.2 *hold, then*

$$\|T_\sigma\|_{S_2} = (2\pi)^{n/2} \left(\int_{\mathbb{C}^n} \|\sigma(z)\|^2_{L^2(\mathbb{C}^n)} dz\right)^{1/2}.$$

Proof Suppose that

$$\int_{\mathbb{C}^n} \|\sigma(z)\|^2_{L^2(\mathbb{C}^n)} dz < \infty.$$

Then there exists a function $h(z)$ in $L^2(\mathbb{C}^n)$ such that

$$\sigma(z) = \rho(z) A_{h(z)} \qquad (3.4)$$

for almost all z in $\mathbb{C}^n$, where $A_{h(z)}$ is the Hilbert–Schmidt operator with kernel $h(z)$. By (6.17) in the book [7], we get for almost all z in $\mathbb{C}^n$,

$$\sigma(z) = \rho(z) W_{\alpha(z)},$$

where

$$\alpha(z) = (2\pi)^{n/2} K^{-1}(h(z))$$

and there is an explicit formula for K^{-1} that is not needed in this proof. All we need to know about K^{-1} is that it is a unitary operator on $L^2(\mathbb{C}^n)$. Thus,

$$\begin{aligned}\int_{\mathbb{C}^n} \|\alpha(z)\|^2_{L^2(\mathbb{C}^n)} dz &= (2\pi)^n \int_{\mathbb{C}^n} \|K^{-1}(h(z))\|^2_{L^2(\mathbb{C}^n)} dz \\ &= (2\pi)^n \int_{\mathbb{C}^n} \|h(z)\|^2_{L^2(\mathbb{C}^n)} dz < \infty.\end{aligned}$$

The converse can be proved as in Theorem 3.1 by reversing the argument and invoking Theorem 2.2. So, by Theorem 3.1, $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a Hilbert–Schmidt operator. By Theorem 3.1, we get

$$\|T_\sigma\|^2_{S_2} = \int_{\mathbb{C}^n} \|\alpha(z)\|^2_{L^2(\mathbb{C}^n)} dz = (2\pi)^n \int_{\mathbb{C}^n} \|h(z)\|^2_{L^2(\mathbb{C}^n)} dz.$$

By (3.4),

$$\|\sigma(z)\|_{S_2} = \|h(z)\|_{L^2(\mathbb{C}^n)}$$

for almost all z in $\mathbb{C}^n$. Therefore

$$\|T_\sigma\|^2_{S_2} = (2\pi)^n \int_{\mathbb{C}^n} \|\sigma(z)\|^2_{S_2} dz$$

and this completes the proof. □

4. **Trace Class Operators**

We give an analog of Theorem 3.1 for trace class pseudo-differential operators for the Weyl transform on $L^2(\mathbb{C}^n)$.

Theorem 4.1. *Let $\sigma : \mathbb{C}^n \to S_2$ be a symbol such that the hypotheses of Theorem 2.2 are satisfied. Then the pseudo-differential operator $T_\sigma : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ is a trace class operator if and only if*

$$\sigma(z) = \rho(z) W_{k(z,\cdot)}, \quad z \in \mathbb{C}^n,$$

where $k \in L^2(\mathbb{C}^n \times \mathbb{C}^n)$ is such that $k^\sharp : \mathbb{C}^n \to L^2(\mathbb{C}^n)$ given by

$$k^\sharp(z)(w) = k(z, w), \quad z, w \in \mathbb{C}^n,$$

is a weakly continuous mapping and there exist functions g and h in $L^2(\mathbb{C}^n \times \mathbb{C}^n)$ for which

$$k = g \circ h.$$

If the conditions for trace class operators hold, then

$$\mathrm{tr}(T_\sigma) = (2\pi)^{-n} \int_{\mathbb{C}^n} (\mathcal{F}_2^{-1} k)(z, z)\, dz,$$

where $\mathcal{F}_2^{-1} k$ denotes the inverse of the partial Fourier transform of k with respect to the "second" variable.

We begin with a lemma.

Lemma 4.1. *Let* $\mathcal{F}_2^{-1}k$ *be as in the preceding theorem. Then*

$$\int_{\mathbb{C}^n} |(\mathcal{F}_2^{-1}k)(z,z)|\,dz < \infty.$$

Proof The starting point is the formula

$$(\mathcal{F}_2^{-1}k)(z,w) = \int_{\mathbb{C}^n} g(z,\zeta)(\mathcal{F}_2^{-1}h)(\zeta,w)\,d\zeta, \quad z,w \in \mathbb{C}^n.$$

Then by Minkowski's inequality in integral form and Plancherel's formula for the Fourier transform, we get

$$\begin{aligned}
&\int_{\mathbb{C}^n} |(\mathcal{F}_2^{-1}k)(z,z)|\,dz \\
= \ & \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} g(z,\zeta)(\mathcal{F}_2^{-1}h)(\zeta,z)|d\zeta \right| dz \\
\leq \ & \int_{\mathbb{C}^n} \left\{ \int_{\mathbb{C}^n} |g(z,\zeta)|\,|(\mathcal{F}_2^{-1}h)(\zeta,z)|\,dz \right\} d\zeta \\
\leq \ & \int_{\mathbb{C}^n} \left\{ \int_{\mathbb{C}^n} |g(z,\zeta)|^2 dz \right\}^{1/2} \left\{ \int_{\mathbb{C}^n} |h(\zeta,z)|^2 dz \right\}^{1/2} d\zeta \\
= \ & \int_{\mathbb{C}^n} \left\{ \int_{\mathbb{C}^n} |g(z,\zeta)|^2 dz \right\}^{1/2} \left\{ \int_{\mathbb{C}^n} |h(\zeta,z)|^2 dz \right\}^{1/2} d\zeta \\
\leq \ & \left\{ \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |g(z,\zeta)|^2 dz\,d\zeta \right\}^{1/2} \left\{ \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |h(\zeta,z)|^2 dz\,d\zeta \right\}^{1/2} \\
= \ & \|g\|_{L^2(\mathbb{C}^n \times \mathbb{C}^n)} \|h\|_{L^2(\mathbb{C}^n \times \mathbb{C}^n)} < \infty.
\end{aligned}$$

□

Proof of Theorem 4.1 As in the proof of Theorem 3.1, we get for all u in $L^2(\mathbb{C}^n)$,

$$(T_\sigma u)(z) = (2\pi)^{-n} \int_{\mathbb{C}^n} k(z,w)\check{u}(w)\,dw, \quad z \in \mathbb{C}^n.$$

Then using the product formula for two Hilbert–Schmidt operators in, say, Theorem 5.2 in [7], we get

$$A_g A_h = A_{g \circ h}.$$

So,

$$(T_\sigma u)(z) = ((A_g A_h)(\check{u}))(z), \quad z \in \mathbb{C}^n.$$

Therefore

$$T_\sigma = A_g A_h \mathcal{F}^{-1} \in S_1.$$

By Lemma 4.1,

$$\mathrm{tr}(T_\sigma) = (2\pi)^{-n} \int_{\mathbb{C}^n} (\mathcal{F}_2^{-1}k)(z,z)\,dz,$$

as claimed. □

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