Pseudo-Differential Operators on the Affine Group

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Abstract. Pseudo-differential operators are defined on the affine group using the Fourier inversion formula for the Fourier transform on the affine group. The Weyl transform on the affine group is given and so are the L^2-L^p estimates for pseudo-differential operators on the affine group.

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1. Introduction

It is a well-known fact from [15] that pseudo-differential operators on $\mathbb{R}^n$ are based on the Plancherel formula for the Fourier transform on $\mathbb{R}^n$. The Plancherel formula gives rise to the Fourier inversion formula, which says that the identity operator for $L^2(\mathbb{R}^n)$ can be expressed in terms of the Fourier transform on $\mathbb{R}^n$. The Fourier inversion formula, albeit useful in many situations, gives a perfect symmetry, namely, the identity operator. By inserting a symbol, which is a suitable function on the phase space $\mathbb{R}^n \times \mathbb{R}^n$, we break the symmetry and obtain a much more interesting and meaningful operator with many applications in sciences and engineering. Such an operator is a pseudo-differential operator on $\mathbb{R}^n$. To extend pseudo-differential operators to other settings, we first observe that $\mathbb{R}^n$ is a group and its dual is also $\mathbb{R}^n$. So, it is natural to extend pseudo-differential operators from $\mathbb{R}^n$ to other groups with explicit dual groups and Fourier inversion formulas for the Fourier transforms on the groups. Such a program has been carried out in some detail for $\mathbb{S}^1$, $\mathbb{Z}$, $\mathbb{Z}_N$, finite abelian groups, compact groups and Heisenberg groups [1, 2, 7, 8, 9, 11] among others.

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The aim of this paper is to move the program forward with the affine group. In Section 2, we recall the basics of the affine group. The Schatten von-Neumann classes that we need to study pseudo-differential operators on affine groups are recalled in Section 3. The Fourier analysis that we need in this paper are given in Section 4. Good references are [5, 13]. In Section 5, we give the Fourier transform on the affine group and show that it is a Weyl transform on $L^2(\mathbb{R})$ [12]. L^2-L^p estimates for pseudo-differential operators on the affine group are given in Sections 6 and 7.

2. The Affine Group

Let U be the upper half plane given by

$$U = \{(b, a) : b \in \mathbb{R},\, a > 0\}.$$

Then we define the binary operation $\cdot$ on U by

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2)$$

for all points (b_1, a_1) and (b_2, a_2) in U. With respect to the multiplication $\cdot$, U is a non-abelian group in which $(0, 1)$ is the identity element and the inverse element of (b, a) is $\left(-\frac{b}{a}, \frac{1}{a}\right)$ for all (b, a) in U. We call U the *affine group*. The left and right Haar measures on U are given by

$$d\mu = \frac{db\, da}{a^2}$$

and

$$d\nu = \frac{db\, da}{a}$$

respectively. Let $H^2_+(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined by

$$H^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [0, \infty)\},$$

where $\operatorname{supp}(\hat{f})$ is the set of every x in $\mathbb{R}$ for which there is no neighborhood of x on which $\hat{f}$ is equal to zero almost everywhere. Similarly, we define $H^2_-(\mathbb{R})$ to be the subspace of $L^2(\mathbb{R})$ by

$$H^2_-(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq (-\infty, 0]\}.$$

Obviously, $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ are closed subspaces of $L^2(\mathbb{R})$.

Let $\pi_\pm : U \to U(H^2_\pm(\mathbb{R}))$ be mappings defined by

$$(\pi_\pm(b, a)f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x - a}{b}\right), \quad x \in \mathbb{R},$$

for all points (b, a) in U and all functions f in $H^2_\pm(\mathbb{R})$. It can be shown that $\pi_\pm : U \to U(H^2_\pm)$ are irreducible and unitary representations of U on $H^2_\pm(\mathbb{R})$.

Details of the affine group and its representations can be found in [5, 13].

For a geometric understanding of the affine group, we look at the set G of all affine mappings given by

$$G = \{T_{b,a} : \mathbb{R} \ni x \mapsto T_{b,a}x = ax + b \in \mathbb{R},\, b \in \mathbb{R}, a > 0\}.$$

G is a group with respect to the composition of mappings. Computing explicitly the composition of the mappings T_{b_1,a_1} and T_{b_2,a_2} in G, we get for all $x \in \mathbb{R}$,

$$\begin{aligned}(T_{b_1,a_1} \circ T_{b_2,a_2})(x) &= T_{b_1,a_1}(T_{b_2,a_2}x) = T_{b_1,a_1}(a_2x + b_2) \\ &= a_1a_2x + b_1 + a_1b_2 = T_{b_1+a_2b_1,a_1a_2}x.\end{aligned}$$

Therefore

$$T_{b_1,a_1} \circ T_{b_2,a_2} = T_{b_1+a_1b_2,a_1a_2}.$$

Thus, the group U is isomorphic to G and this is precisely the justification for calling U the affine group.

We can give another way to look at the affine group. The set $\mathbb{R}$ of all positive numbers is clearly an additive group isomorphic to the group $\{T_{b,1} : b \in \mathbb{R}\}$ of translations, which we denote by N. That N is a normal subgroup of G is easy to check. The set $\mathbb{R}^+$ of all positive real numbers is a group with respect to multiplication and is isomorphic to the group $\{T_0, a : a > 0\}$ of dilations, which we denote by A. Since $N \cap A = \{T_{1,0}\}$, it follows that the affine group G is given by

$$G = AN,$$

and we call G the *internal semi-direct product* of A and N and we write

$$G = A \ltimes N$$

or

$$G = \mathbb{R}^+ \ltimes \mathbb{R}.$$

More information about semi-direct products can be found in Section 5.5 of the book [4].

It should also be mentioned that the affine group is closely related to the special linear group SL(2, $\mathbb{R}$) given by

$$\mathrm{SL}\,(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\, ad - bc \neq 0 \right\}.$$

By the Iwasawa decomposition, we can write

$$\mathrm{SL}\,(2,\mathbb{R}) = KAN,$$

where

$$K = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\},$$

$$A = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{bmatrix} : \alpha > 0 \right\}$$

and

$$N = \left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} : \beta \in \mathbb{R} \right\}.$$

The group AN is in fact the affine group. See [6] and page 136 of the book [14].

3. Schatten-von Neumann Classes

Let X be an infinite-dimensional, separable and complex Hilbert space with inner product $(\,,\,)_X$ and norm $\|\,\|_X$. Let $A : X \to X$ be a compact operator. Then $\sqrt{A^*A} : X \to X$ is a positive and compact operator. Hence the spectral theorem gives an orthonormal basis $\{\varphi_k : k = 1, 2, \dots\}$ for X consisting of eigenvectors of $\sqrt{A^*A}$. For $k = 1, 2, \dots,$ let s_k be the eigenvalue of $\sqrt{A^*A}$ corresponding to the eigenvector φ_k. Then for $1 \le p < \infty$, we say that A is in the *Schatten von-Neumann class* S_p if

$$\sum_{k=1}^{\infty} s_k^p < \infty.$$

If $A \in S_p$, then the Schatten-von Neumann norm $\|A\|_{S_p}$ of A is defined by

$$\|A\|_{S_p} = \left(\sum_{k=1}^{\infty} s_k^p \right)^{1/p}.$$

By convention, S_∞ is taken to be the C^*-algebra of all bounded linear operators on X and the norm in S_∞ is simply the operator norm $\|\,\|_*$.

4. Fourier Analysis on the Affine Group

We give in this section the Fourier analysis on the affine group emphasizing the Fourier transform, the Plancherel formula and the Fourier inversion formula. To this end, we find it convenient to reformulate the irreducible and unitary representations of the affine group U on $U(H^2_\pm(\mathbb{R}))$. Let

$$\mathbb{R}_+ = [0, \infty)$$

and

$$\mathbb{R}_- = (-\infty, 0].$$

Then we look at the equivalents of $\pi_+ : U \to U(H^2_+(\mathbb{R}))$ and $\pi_- : U \to U(H^2_-(\mathbb{R}))$ denoted by, respectively, $\rho_+ : U \to U(L^2(\mathbb{R}_+))$ and $\rho_- : U \to U(L^2(\mathbb{R}_-))$, and given by

$$(\rho_+(b, a)u)(s) = a^{1/2} e^{-ibs} u(as), \quad s \in \mathbb{R}_+,$$

for all $u \in L^2(\mathbb{R}_+)$, and

$$(\rho_-(b, a)v)(s) = a^{1/2} e^{-ibs} v(as), \quad s \in \mathbb{R}_-,$$

for all $v \in L^2(\mathbb{R}_-)$. For all $\varphi \in L^2(\mathbb{R}_\pm)$, we define the functions $D_\pm \varphi$ on $\mathbb{R}_\pm$ by

$$(D_\pm \varphi)(s) = |s|^{1/2} \varphi(s), \quad s \in \mathbb{R}_\pm.$$

The unbounded linear operators $D_\pm$ on $L^2(\mathbb{R}_\pm)$ are known as the *Duflo–Moore operators* [3].

Let $f \in L^2(U)$. Then we define the Fourier transform $\hat{f}$ on $\{\rho_+, \rho_-\}$ by

$$(\hat{f}(\rho_\pm)\varphi)(s) = \int_0^\infty \int_{-\infty}^\infty f(b,a)(\rho_\pm(b,a)D_\pm\varphi)(s)\,\frac{db\,da}{a^2}, \quad s \in \mathbb{R}_\pm,$$

for all $\varphi \in L^2(\mathbb{R}_\pm)$. We have the Plancherel formula to the effect that

$$\|\hat{f}(\rho_+)\|_{S_2}^2 + \|\hat{f}(\rho_-)\|_{S_2}^2 = \|f\|_{L^2(U)}^2,$$

where $\|\,\|_{S_2}$ is the norm in the Hilbert space S_2 of all Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

The Fourier inversion formula states that for all $f \in L^2(U)$, we get

$$f(b,a) = \mathrm{tr}(\hat{f}(\rho_+)\rho_+(b,a)^*) + \mathrm{tr}(\hat{f}(\rho_-)\rho_-(b,a)^*)$$

for all $(b,a) \in U$.

We find it convenient to denote $\{\rho_+, \rho_-\}$ by $\{\pm\}$. Let $\sigma : U \times \{\pm\} \to B(L^2(\mathbb{R}))$ be a mapping, where $B(L^2(\mathbb{R}))$ is the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R})$. Then for all $f \in L^2(U)$, we define $T_\sigma f$ *formally* to be the function on U by

$$(T_\sigma f)(b,a) = \sum_{j=\pm} \mathrm{tr}\,(\hat{f}(\rho_j)\rho_j(b,a)^*\sigma(b,a,j)), \quad (b,a) \in U.$$

We call T_σ the pseudo-differential operator on the affine group U corresponding to the operator-valued symbol σ.

5. The Fourier Transform on the Affine Group

Let $f \in L^2(U)$. Then for all $\varphi \in L^2(\mathbb{R}_+)$, we get for all $s \in (0,\infty)$,

$$(\hat{f}(\rho_+)\varphi)(s) = \int_{-\infty}^\infty \int_0^\infty f(b,a)a^{1/2}e^{-ibs}(as)^{1/2}\varphi(as)\frac{da\,db}{a^2}.$$

Let $as = t$. Then $da = \frac{dt}{s}$ and we have

$$\begin{aligned}
&(\hat{f}(\rho_+)\varphi)(s) \\
= \;& \int_{-\infty}^\infty \int_0^\infty f\left(b, \frac{t}{s}\right)\left(\frac{t}{s}\right)^{1/2} e^{-ibs}t^{1/2}\varphi(t)s\frac{dt\,db}{t^2} \\
= \;& \int_{-\infty}^\infty \int_0^\infty f\left(b, \frac{t}{s}\right) s^{1/2}e^{-ibs}\varphi(t)\frac{dt\,db}{t} \\
= \;& \int_0^\infty K_+^f(s,t)\varphi(t)\,dt,
\end{aligned}$$

where

$$K_+^f(s,t) = \frac{\sqrt{s}}{t}\int_{-\infty}^\infty f\left(b, \frac{t}{s}\right)e^{-ibs}db = \frac{\sqrt{s}}{t}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s, \frac{t}{s}\right)$$

for $0 < s, t < \infty$, where $\mathcal{F}_1 f$ denotes the Fourier transform of f with respect to the first variable.

Similarly, for all $\varphi \in L^2(\mathbb{R}_-)$, we obtain for all $s \in (-\infty, 0)$,

$$(\hat{f}(\rho_-)\varphi)(s) = \int_{-\infty}^{0} K_-^f(s,t)\varphi(t)\,dt,$$

where

$$K_-^f(s,t) = \frac{\sqrt{|s|}}{|t|}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s, \frac{t}{s}\right)$$

for $-\infty < s, t < 0$.

Let $f \in L^2(U)$. Then we define the bounded linear operator $\hat{f}(\rho) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\hat{f}(\rho)\varphi = \hat{f}(\rho_+)\varphi_+ + \hat{f}(\rho_-)\varphi_-,$$

where

$$\varphi_\pm = \varphi\chi_{\mathbb{R}_\pm}.$$

Here,

$$\chi_{\mathbb{R}_\pm}(s) = \begin{cases} 1, & s \in \mathbb{R}_\pm, \\ 0, & s \notin \mathbb{R}_\pm. \end{cases}$$

Thus, we have the following result.

Theorem 5.1. *Let $f \in L^2(U)$. Then for all $\varphi \in L^2(\mathbb{R})$,*

$$(\hat{f}(\rho)\varphi)(s) = \int_{-\infty}^{\infty} K^f(s,t)\,\varphi(t)\,dt, \quad s \in \mathbb{R},$$

where

$$K^f(s,t) = \begin{cases} K_+^f(s,t), & s > 0,\ t > 0, \\ K_-^f(s,t), & s < 0,\ t < 0, \\ 0, & s > 0,\ t < 0, \\ 0, & s < 0,\ t > 0. \end{cases} \tag{5.1}$$

That the Fourier transform on the affine group is a Weyl transform on $L^2(\mathbb{R})$ is the content of the following theorem. First we recall the twisting operator T in [12] given by

$$(Tf)(x,y) = f\left(x + \frac{y}{2}, x - \frac{y}{2}\right), \quad x, y \in \mathbb{R},$$

for all measurable functions f on $\mathbb{R} \times \mathbb{R}$.

Theorem 5.2. *Let $f \in L^2(U)$. Then for all $\varphi \in L^2(\mathbb{R})$,*

$$\hat{f}(\rho)\varphi = W_{\sigma_f}\varphi, \quad \varphi \in L^2(\mathbb{R}),$$

where

$$\sigma_f(x,\xi) = (2\pi)^{-1/2}(\mathcal{F}_2 T K^f)(x,\xi), \quad x, \xi \in \mathbb{R}.$$

6. L^2 Boundedness

Theorem 6.1. *Let $\sigma : U \times \{\pm\} \to S_p$ be such that*

$$\sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_p}^2 \frac{db\,da}{a^2} < \infty,$$

where $1 \le p \le 2$. Then $T_\sigma : L^2(U) \to L^2(U)$ is a bounded linear operator. Moreover,

$$\|T_\sigma\|_* \le \left\{ \sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_p}^2 \frac{db\,da}{a^2} \right\}^{1/2},$$

where $\| \, \|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(U)$. Then using Minkowski's inequality in integral form, we get

$$\begin{aligned} &\|T_\sigma f\|_{L^2(U)} \\ = &\left\{ \int_0^\infty \int_{-\infty}^\infty |(T_\sigma f)(b,a)|^2 \frac{db\,da}{a^2} \right\}^{1/2} \\ = &\left\{ \int_0^\infty \int_{-\infty}^\infty \left| \sum_{j=\pm} \mathrm{tr}\,(\hat{f}(\rho_j)\rho_j(b,a)^* \sigma(b,a,j)) \right|^2 \frac{db\,da}{a^2} \right\}^{1/2} \\ \le &\sum_{j=\pm} \left\{ \int_0^\infty \int_{-\infty}^\infty |\mathrm{tr}\,(\hat{f}(\rho_j)\rho_j(b,a)^* \sigma(b,a,j))|^2 \frac{db\,da}{a^2} \right\}^{1/2}. \end{aligned} \tag{6.1}$$

For $1 \le p \le q \le \infty$, it follows from the definition of the Schatten-von Neumann classes that

$$S_p \subseteq S_q$$

and

$$\|A\|_{S_q} \le \|A\|_{S_p}, \quad A \in S_p.$$

Thus, it follows from (6.1) that

$$
\begin{aligned}
&\|T_\sigma f\|_{L^2(U)}\\
\leq\ & \sum_{j=\pm}\left\{\int_0^\infty\int_{-\infty}^\infty \|\hat{f}(\rho_j)\|_{S_2}^2\|\sigma(b,a,j)\|_{S_2}^2\frac{db\,da}{a^2}\right\}^{1/2}\\
\leq\ & \sum_{j=\pm}\left\{\int_0^\infty\int_{-\infty}^\infty \|\hat{f}(\rho_j)\|_{S_2}^2\|\sigma(b,a,j)\|_{S_p}^2\frac{db\,da}{a^2}\right\}^{1/2}\\
=\ & \sum_{j=\pm}\|\hat{f}(\rho_j)\|_{S_2}\left\{\int_0^\infty\int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_p}^2\frac{db\,da}{a^2}\right\}^{1/2}\\
\leq\ & \left\{\sum_{j=\pm}\|\hat{f}(\rho_j)\|_{S_2}^2\right\}^{1/2}\left\{\sum_{j=\pm}\int_0^\infty\int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_p}^2\frac{db\,da}{a^2}\right\}^{1/2}\\
=\ & \left\{\sum_{j=\pm}\int_0^\infty\int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_p}^2\frac{db\,da}{a^2}\right\}^{1/2}\|f\|_{L^2(U)}.
\end{aligned}
$$

□

7. L^2-L^p Estimates, $2 \leq p \leq \infty$

Theorem 7.1. *Let* $\sigma : U \times \{\pm\} \to S_{p'}$ *be such that*

$$\sum_{j=\pm}\int_0^\infty\int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_{p'}}^p\frac{db\,da}{a^2} < \infty,$$

where $2 \leq p < \infty$ *and* p' *is the conjugate index of* p*, i.e.,* $\frac{1}{p}+\frac{1}{p'} = 1$*. Then* $T_\sigma : L^2(U) \to L^p(U)$ *is a bounded linear operator and*

$$\|T_\sigma\|_{B(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))} \leq \left\{\left[\sum_{j=\pm}\int_0^\infty\int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_{p'}}^p\frac{db\,da}{a^2}\right]^{2/p}\right\}^{1/2},$$

where $\|\ \|_{B(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))}$ *is the norm in the Banach space of all bounded linear operators from* $L^2(\mathbb{R}^n)$ *into* $L^p(\mathbb{R}^n)$*.*

Proof. Let $f \in L^p(U)$. Then using Minkowski's inequality in integral form, we get

$$\begin{aligned}
&\|T_\sigma f\|_{L^p(U)} \\
= &\left\{\int_0^\infty \int_{-\infty}^\infty |(T_\sigma f)(b,a)|^p \frac{db\,da}{a^2}\right\}^{1/p} \\
= &\left\{\int_0^\infty \int_{-\infty}^\infty \left|\sum_{j=\pm} \operatorname{tr}\left(\hat{f}(\rho_j)\rho_j(b,a)^*\sigma(b,a,j)\right)\right|^p \frac{db\,da}{a^2}\right\}^{1/p} \\
\leq &\sum_{j=\pm}\left\{\int_0^\infty \int_{-\infty}^\infty |\operatorname{tr}\left(\hat{f}(\rho_j)\rho_j(b,a)^*\sigma(b,a,j)\right)|^p \frac{db\,da}{a^2}\right\}^{1/p}. \qquad (7.1)
\end{aligned}$$

Now, using Hölder's inequality and the Plancherel theorem, it follows from (7.1) that

$$\begin{aligned}
&\|T_\sigma f\|_{L^p(U)} \\
\leq &\sum_{j=\pm}\left\{\int_0^\infty \int_{-\infty}^\infty \|\hat{f}(\rho_j)\|_{S_p}^p \|\sigma(b,a,j)\|_{S_{p'}}^p \frac{db\,da}{a^2}\right\}^{1/p} \\
= &\sum_{j=\pm}\|\hat{f}(\rho_j)\|_{S_p}\left\{\int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_{p'}}^p \frac{db\,da}{a^2}\right\}^{1/p} \\
\leq &\sum_{j=\pm}\|\hat{f}(\rho_j)\|_{S_2}\left\{\int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_{p'}}^p \frac{db\,da}{a^2}\right\}^{1/p} \\
= &\left\{\sum_{j=\pm}\|\hat{f}(\rho_j)\|_{S_2}^2\right\}^{1/2}\left\{\left[\sum_{j=\pm}\int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)\|_{S_{p'}}^p \frac{db\,da}{a^2}\right]^{2/p}\right\}^{1/2}
\end{aligned}$$

and this completes the proof. ∎

Remark 7.2. The conclusion of Theorem 7.1 can be expressed in the form

$$\|T_\sigma\|_{B(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))} \leq \left\|\|\|\sigma(\cdot,\cdot,j)\|_{S_{p'}}\|_{L^p(U)}\right\|_{l^2(\pm)}, \quad 2 \leq p < \infty.$$

We fill in the endpoint $p = \infty$ in the following theorem.

Theorem 7.3. *Let $\sigma : U \times \{\pm\} \to S_1$ be such that*

$$\left\|\|\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U)}\right\|_{l^2(\pm)} < \infty.$$

Then $T_\sigma : L^2(U) \to L^\infty(U)$ is a bounded linear operator and

$$\|T_\sigma\|_{B(L^2(\mathbb{R}^n),L^\infty(\mathbb{R}^n))} \leq \left\|\|\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U)}\right\|_{l^2(\pm)}.$$

Proof. Let $f \in L^\infty(U)$. Then by Minkowski's inequality,

$$\begin{aligned}
&\|T_\sigma f\|_{L^\infty(U)} \\
&= \left\| \sum_{j=\pm} \operatorname{tr}\left(\hat{f}(\rho_j)\rho_j(\cdot,\cdot)^*\sigma(\cdot,\cdot,j)\right) \right\|_{L^\infty(U)} \\
&\leq \sum_{j=\pm} \|\operatorname{tr}\left(\hat{f}(\rho_j)\rho_j(\cdot,\cdot)^*\sigma(\cdot,\cdot,j)\right)\|_{L^\infty(U)}.
\end{aligned} \tag{7.2}$$

Using Hölder's inequality and Plancherel's theorem, it follows from (7.2) that

$$\begin{aligned}
&\|T_\sigma f\|_{L^\infty(U)} \\
&\leq \sum_{j=\pm} \left\| \|\hat{f}(\rho_j)\|_{S_\infty} \|\sigma(\cdot,\cdot,j)\|_{S_1} \right\|_{L^\infty(U)} \\
&= \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_\infty} \, \|\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U)} \\
&\leq \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2} \, \|\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U)} \\
&\leq \left\{ \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2}^2 \right\}^{1/2} \left\{ \sum_{j=\pm} \|\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U)}^2 \right\}^{1/2} \\
&= \left\{ \sum_{j=\pm} \|\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U)}^2 \right\}^{1/2} \|f\|_{L^2(U)}.
\end{aligned}$$

□

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