The Dirichlet Divisor Problem, Traces and Determinants for Complex Powers of the Twisted Bi-Laplacian

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Abstract Estimating the counting function for the eigenvalues of the twisted bi-Laplacian leads to the Dirichlet divisor problem, which is then used to compute the trace of the heat semigroup and the Dixmier trace of the inverse of the twisted bi-Laplacian. The zeta function regularizations of the traces and determinants of complex powers of the twisted bi-Laplacian are computed. A formula for the zeta function regularizations of determinants of heat semigroups of complex powers of the twisted bi-Laplacian is given.

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1 Introduction

The twisted Laplacian L on $\mathbb{R}^2$ is the second-order partial differential operator given by

$$L = -\Delta + \frac{1}{4}(x^2 + y^2) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Thus, the twisted Laplacian L is the Hermite operator

$$H = -\Delta + \frac{1}{4}(x^2 + y^2)$$

perturbed by the partial differential operator $-iN$, where

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation operator.

That H is called the Hermite operator is due to the fact that Hermite functions are the eigenfunctions of H. See, for instance, Section 6.4 in [9]. That N is called the rotation operator can be attributed to the fact that in polar coordinates,

$$N = \frac{\partial}{\partial \theta},$$

which is the simplest differential operator on the unit circle centered at the origin.

The twisted Laplacian appears in harmonic analysis naturally in the context of Wigner transforms and Weyl transforms [2, 12]. In the paper [1], it is shown that L is essentially self-adjoint, and the spectrum $\Sigma(L_0)$ of the closure L_0 is given by a sequence of eigenvalues, which are odd natural numbers, *i.e.*,

$$\Sigma(L_0) = \{2k + 1 : k = 0, 1, 2, \dots\}.$$

It should be noted, however, that each eigenvalue has infinite multiplicity.

Renormalizing the twisted Laplacian L to the partial differential operator P given by

$$P = \frac{1}{2}(L+1), \tag{1.2}$$

we see that the eigenvalues of P are the natural numbers $1, 2, \ldots$, and each eigenvalue, as in the case of L, has infinite multiplicity.

Now, the conjugate $\overline{L}$ of the twisted Laplacian L is given by

$$\overline{L} = -\Delta + \frac{1}{4}(x^2+y^2) + i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{1.3}$$

and after renormalization, we get the conjugate Q of P given by

$$Q = \frac{1}{2}(\overline{L}+1). \tag{1.4}$$

The aim of this paper is to analyze the heat kernels and Green functions of complex powers of the twisted bi-Laplacian M defined by

$$M = QP = PQ = \frac{1}{4}(H - iN + 1)(H + iN + 1), \tag{1.5}$$

where P and Q commute because it can be shown by easy computations that H and N commute, i.e., $HNf = NHf$ for all functions f in $C^\infty(\mathbb{R}^2)$.

It is proved in [3] that M is essentially self-adjoint on $L^2(\mathbb{R}^2)$. The unique self-adjoint extension of M on $L^2(\mathbb{R}^2)$ is again denoted by M.

In order to describe the spectral properties of M precisely, let us first recall that the Fourier–Wigner transform $V(f, g)$ of two functions f and g in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ is the function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ given by

$$V(f,g)(q,p) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iqy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy$$

for all q and p in $\mathbb{R}$. For $k = 0, 1, 2, \ldots$, the Hermite function e_k of order k is defined on $\mathbb{R}$ by

$$e_k(x) = \frac{1}{(2^k k!\sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k e^{-x^2}, \quad x \in \mathbb{R}. \tag{1.7}$$

Now, for $j, k = 0, 1, 2, \ldots,$ we define the function $e_{j,k}$ on $\mathbb{R}^2$ by

$$e_{j,k}(x, y) = V(e_j, e_k)(x, y), \quad x, y \in \mathbb{R}. \tag{1.8}$$

It can be shown that $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$. See, for example, Theorem 21.2 in [12].

The following result is Theorem 1.1 in [3].

Theorem 1.1 *The eigenvalues and the eigenfunctions of the twisted bi-Laplacian M are, respectively, the natural numbers $1, 2, 3, \ldots,$ and the functions $e_{j,k}$, $j, k = 0, 1, 2, \ldots$. More precisely, for $n = 1, 2, 3, \ldots,$ the eigenfunctions corresponding to the eigenvalue n are all the functions $e_{j,k}$ where $j, k = 0, 1, 2, \ldots,$ such that*

$$(j + 1)(k + 1) = n.$$

By means of Theorem 1.1, we see that the multiplicity of each eigenvalue n of the twisted bi-Laplacian is equal to the number $d(n)$ of divisors of the positive integer n. We give as Corollary 1.2 in [3] an estimate on the counting function $N(\lambda)$ defined as the number of eigenvalues of M less than or equal to λ. In fact, we can see that the following result, which is Corollary 1.2 in [3], is the well-known result on asymptotic behavior of the Dirichlet divisors in the perspective of the counting function of the twisted bi-Laplacian, in which the multiplicity of each eigenvalue is taken into account.

Theorem 1.2 *For all λ in $[0, \infty)$,*

$$N(\lambda) = \sum_{n \leq \lambda} d(n) = \lambda \ln \lambda + (2\gamma - 1)\lambda + E(\lambda), \tag{1.9}$$

where γ is Euler's constant and

$$E(\lambda) = O(\sqrt{\lambda})$$

as $\lambda \to \infty$.

Remark 1.3 More precise results than Theorems 1.1 and 1.2 can be found in [4]. A complete and classical proof of Theorem 1.2 can be based on Theorem 3.12 in [10] and the above-mentioned connection between the Dirichlet divisors and the twisted bi-Laplacian. It is interesting to point out the connection with the Dirichelet divisor problem, which asks for the best number μ such that

$$E(\lambda) = O(\lambda^\mu)$$

as $\lambda \to \infty$. The conjecture is that $\mu = 1/4$, but it is a result of Hardy [5] that $\mu = 1/4$ does not work. The best result to date seems to be due to Soundararajan [8].

Theorem 1.2 is used to compute the trace of the heat semigroup of M in Section 2 and the Dixmier trace of the inverse of M in Section 3. Another theme of this paper is to compute the zeta function regularizations of the trace and the determinant of the complex power M^α of M, where $\alpha \in \mathbb{C}$. To that end, we use the complex-valued function ζ_{M^α} defined *formally* by

$$\zeta_{M^\alpha}(s) = \mathrm{tr}((M^\alpha)^{-s}) = \mathrm{tr}(M^{-\alpha s}), \quad s \in \mathbb{C},$$

in Section 4 to compute the zeta function regularizations of the trace and determinant of M^α, and give a formula for the zeta function regularization of the determinant of the heat semigroup e^{-tM^α}.

2 The Trace of the Heat Semigroup

Theorem 2.1 *For* $t > 0$,

$$\mathrm{tr}(e^{-tM}) = (\gamma - \ln t)t^{-1} + O(t^\mu),$$

where $\mu > \frac{1}{4}$.

Proof Since

$$\mathrm{tr}(e^{-tM}) = \int_0^\infty e^{-t\lambda} dN(\lambda),$$

it follows from an integration by parts that for $t > 0$,

$$\mathrm{tr}(e^{-tM}) = e^{-t\lambda}N(\lambda)\big|_0^\infty + t\int_0^\infty e^{-t\lambda}N(\lambda)d\lambda = t\int_0^\infty e^{-t\lambda}N(\lambda)\,d\lambda. \quad (2.1)$$

So, using the formula for $N(\lambda)$ in Section 1 and (2.1), we get for $t > 0$,

$$\begin{aligned}\operatorname{tr}(e^{-tM}) &= t\int_0^\infty e^{-t\lambda}(\lambda\ln\lambda + (2\gamma-1)\lambda + O(\lambda^\mu))\,d\lambda \\ &= t\int_0^\infty e^{-t\lambda}\lambda\ln\lambda\,d\lambda + (2\gamma-1)t^{-1} + O(t^\mu). \qquad (2.2)\end{aligned}$$

Since

$$\begin{aligned}\int_0^\infty e^{-t\lambda}\lambda\ln\lambda\,d\lambda &= -\frac{d}{dt}\int_0^\infty e^{-t\lambda}\ln\lambda\,d\lambda = \frac{d}{dt}\left[\frac{1}{t}(\gamma+\ln t)\right] \\ &= (1-\gamma-\ln t)t^{-2}, \qquad (2.3)\end{aligned}$$

it follows from (2.2) and (2.3) that for $t > 0$,

$$\operatorname{tr}(e^{-tM}) = (\gamma - \ln t)t^{-1} + O(t^\mu),$$

as required.

$\square$

3 The Dixmier Trace of the Inverse

We first begin with a version of the Dixmier trace that is tailored for the inverse of the twisted bi-Laplacian M. The book [7] is a comprehensive account of Dixmier traces and related topics. In particular, Chapter 1 of the book [7] contains motivational and background material on Dixmier traces.

Let A be a positive and compact operator on a complex and separable Hilbert space X. Let

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots$$

be the eigenvalues of A arranged in decreasing order with multiplicities counted. For a positive integer k, we say that A is in the k^{th} Dixmier trace class if

$$\left\{\frac{1}{\ln^k N}\sum_{j=1}^N \lambda_j(A)\right\}_{N=2}^\infty \in l^\infty.$$

If A is in the k^{th} Dixmier trace class such that $\lim_{N\to\infty} \frac{1}{\ln^k N} \sum_{j=1}^{N} \lambda_j(A)$ exists, then the k^{th} Dixmier trace $\mathrm{tr}_k(A)$ of A is given by

$$\mathrm{tr}_k(A) = \lim_{N\to\infty} \frac{1}{\ln^k N} \sum_{j=1}^{N} \lambda_j(A).$$

Using Theorem 1.2, we get the following theorem for the Dixmier trace of M^{-1}.

Theorem 3.1 *M^{-1} is in the second Dixmier trace class and*

$$\mathrm{tr}_2(M^{-1}) = \frac{1}{2}.$$

Proof Let us compute $\sum_{n\le x} \frac{d(n)}{n}$ for large and positive integers x, say, for $x > 2$. To do this, we use the partial summation formula to the effect that

$$\sum_{n\le x} a_n f(n) = S(x-1)f(x) - \int_1^x S(t)f'(t)\,dt, \tag{3.1}$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence with positive terms, f is a positive and differentiable function on $(0,\infty)$, and S is the function on $[1,\infty)$ given by

$$S(t) = \sum_{n\le t} a_n, \quad t \ge 1. \tag{3.2}$$

Indeed,

$$\begin{aligned}
\int_1^x S(t)f'(t)\,dt &= \sum_{n=1}^{x-1} \int_n^{n+1} S(t)f'(t)\,dt \\
&= \sum_{n=1}^{x-1} \int_n^{n+1} \left(\sum_{k=1}^{n} a_k\right) f'(t)\,dt \\
&= \sum_{n=1}^{x-1} \sum_{k=1}^{n} a_k(f(n+1) - f(n)).
\end{aligned}$$

Interchanging the order of summation, we get

$$\begin{aligned}\int_1^x S(t)f'(t)\,dt &= \sum_{k=1}^{x-1}\sum_{n=k}^{x-1} a_k(f(n+1)-f(n))\\ &= \sum_{k=1}^{x-1} a_k(f(x)-f(k)).\end{aligned}$$

Therefore

$$S(x-1)f(x)-\int_1^x S(t)f'(t)\,dt = \sum_{n=1}^{x} a_n f(n),$$

which is (3.1). Applying (3.1) and (3.2) with $a_n = d(n)$ and $f(n) = \frac{1}{n}$, and using the asymptotic formula for the function S as given by the Dirichlet divisor problem, we get

$$\begin{aligned}\sum_{n\le x}\frac{d(n)}{n} &= S(x-1)f(x)-\int_1^x S(t)f'(t)\,dt\\ &= \frac{1}{x}((x-1)\ln(x-1)+(2\gamma-1)(x-1)+O(\sqrt{x}))\\ &\quad+\int_1^x\left(\frac{\ln t}{t}+(2\gamma-1)t^{-1}+O(t^{-3/2})\right)\,dt. \qquad (3.3)\end{aligned}$$

Since

$$(x-1)\ln(x-1) = x\ln x + O(\sqrt{x}) \qquad (3.4)$$

as $x\to\infty$, and

$$\int_1^x \frac{\ln t}{t}dt = \frac{1}{2}\ln^2 x. \qquad (3.5)$$

it follows from (3.3)–(3.5) that

$$\begin{aligned}\sum_{n\le x}\frac{d(n)}{n} &= \frac{1}{x}(x\ln x+(2\gamma-1)x+O(\sqrt{x}))\\ &\quad+\frac{1}{2}\ln^2 x+(2\gamma-1)\ln x+O(x^{-1/2})\\ &= \frac{1}{2}\ln^2 x+2\gamma\ln x+(2\gamma-1)+O(x^{-1/2})\end{aligned}$$

as $x\to\infty$. This completes the proof. $\square$

4 Zeta Function Regularizations

We begin with the following easy observation.

Lemma 4.1 *Let $\alpha \in \mathbb{C}$. Then for all s with* $\mathrm{Re}\,(\alpha s) > 1$,

$$\zeta_{M^\alpha}(s) = \zeta^2(\alpha s).$$

Proof Let $s \in \mathbb{C}$ be such that $\mathrm{Re}\,(\alpha s) > 1$. Then by Theorem 1.1, the eigenvalues of $M^{-\alpha s}$ are $n^{-\alpha s}$, $n = 1, 2, \ldots$, and the multiplicity of $n^{-\alpha s}$ is equal to the number $d(n)$ of Dirichlet divisors of n. Therefore

$$\zeta_{M^\alpha}(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^{\alpha s}}. \tag{4.1}$$

So, a straightforward computation gives

$$\zeta^2(\alpha s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^{\alpha s}} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{\alpha s}} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha s}} \sum_{\mu\nu=n} 1 = \sum_{n=1}^{\infty} \frac{d(n)}{n^{\alpha s}}.$$

□

The zeta function regularizations of the trace and the determinant of M^α, denoted by $\mathrm{tr}_R(M^\alpha)$ and $\det_R(M^\alpha)$ respectively, are defined by

$$\mathrm{tr}_R(M^\alpha) = \zeta_{M^\alpha}(-1)$$

and

$$\det_R(M^\alpha) = e^{-\zeta'_{M_\alpha}(0)}.$$

The physical meanings of these quantities can be found in, e.g., the paper [6].

Theorem 4.2 *Let $\alpha \in \mathbb{C} \setminus \{-1\}$. Then*

$$\mathrm{tr}_R(M^\alpha) = \zeta^2(-\alpha).$$

Proof By Lemma 4.1 and the analytic continuation of the Riemann zeta function to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, we see that

$$\mathrm{tr}_R(M^\alpha) = \zeta_{M^\alpha}(-1) = \zeta^2(-\alpha).$$

□

Remark 4.3 It is well-known from, say, [11] that

$$\zeta(-1) = -\frac{1}{12}.$$

Hence

$$\mathrm{tr}_R(M) = \frac{1}{144}.$$

Remark 4.4 In the case when $\alpha = -1$, the zeta function regularization of the inverse M^{-1} is equal to infinity. The Dixmier trace instead of the trace of the inverse M^{-1} is a finite number.

Theorem 4.5 *Let $\alpha \in \mathbb{C}$. Then*

$$\det_R(M^\alpha) = (2\pi)^{-\alpha/2}.$$

Proof As in Theorem 4.2,

$$\det_R(M^\alpha) = e^{-\zeta'_{M^\alpha}(0)} = e^{-2\alpha\zeta(0)\zeta'(0)}.$$

It can be found in [11] again that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$. So,

$$\det_R(M^\alpha) = (2\pi)^{-\alpha/2}.$$

□

As an application, we can give a formula for the determinants of the heat semigroups of complex powers of the twisted bi-Laplacian.

Theorem 4.6 *Let $\alpha \in \mathbb{C} \setminus \{-1\}$. Then for $t > 0$,*

$$\det_R(e^{-tM^\alpha}) = e^{-t\zeta^2(-\alpha)}.$$

Proof By Theorem 1.1, the eigenvalues of $e^{(-tM^\alpha)^{-s}}$ are $e^{tn^\alpha s}$, $n = 1, 2, \ldots,$ and the multiplicity of the eigenvalue $e^{tn^\alpha s}$ is $d(n)$. Therefore

$$\zeta_{e^{-tM^\alpha}}(s) = \mathrm{tr}((e^{-tM^\alpha})^{-s}) = \sum_{n=1}^{\infty} d(n)e^{tn^\alpha s}, \quad s \in \mathbb{C}.$$

So, by equation (4.1) and Theorem 4.2,

$$\zeta'_{e^{-tM^\alpha}}(0) = t\sum_{n=1}^{\infty} d(n)n^\alpha = t\zeta^2(-\alpha).$$

Thus,

$$\det_R(e^{-tM^\alpha}) = e^{-\zeta'_{e^{-tM^\alpha}}(0)} = e^{-t\zeta^2(-\alpha)},$$

and this completes the proof. $\square$

Remark 4.7 By Theorems 4.2 and 4.6, we see that for $\alpha \in \mathbb{C} \setminus \{-1\}$,

$$\det_R(e^{-tM^\alpha}) = e^{-t\mathrm{tr}_R(M^\alpha)}, \quad t > 0,$$

which is in conformity with the well-known relationship between the determinant and the trace of a square matrix A given by

$$\det(e^A) = e^{\mathrm{tr}\,(A)}.$$

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