Mathematics 1300 3.0 AF
Solutions to Test 1

1. Solution: (i) Let $x \to 0$ along the numbers $\frac{1}{\left(n+\frac{1}{2}\right)\pi}$. Then

$$\sin\frac{1}{x} = \sin\left(n+\frac{1}{2}\right)\pi = \sin n\pi\cos\frac{\pi}{2} + \cos n\pi\sin\frac{\pi}{2} = (-1)^n.$$

So, $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

Solution: (ii) Since

$$-x^2 \le x^2\sin\frac{1}{x} \le x^2$$

and $\lim_{x\to 0} x^2 = 0$, it follows from the squeeze theorem that

$$\lim_{x\to 0} x^2\sin\frac{1}{x} = 0.$$

2. Solution: $|f(x)-5| < 0.001$ if $|(2x+1)-5| < 0.001$ or if $|2x-4| < 0.001$ or if $2|x-2| < 0.001$ or if $|x-2| < 0.0005$. So, let δ be any positive number ≤ 0.0005. Then

$$0 < |x-2| < \delta \Rightarrow |f(x)-5| = 2|x-2| < 2\delta = 2(0.0005) = 0.001.$$

3. Solution: Let $f(x) = \cos x - x$. Then $f(0) = 1$ and $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$ Therefore 0 is between $f(0)$ and $f\left(\frac{\pi}{2}\right)$ By the Intermediate Value Theorem, there exists a number c in $\left(0, \frac{\pi}{2}\right)$ such that

$$f(c) = 0.$$

Therefore

$$\cos c = c.$$

4. Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

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\begin{aligned}
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
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The domain of f is $[0, \infty)$ and the domain of f' is $(0, \infty)$.